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ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1934 JUNE 8

Professor F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

George William Evans-Cross, M.A., D.Sc., Avenue House, 20 Grand Avenue, Bournemouth East ; and

Alfred Percy Freeman, Sunnylands, High Road, Hutton, Brentwood, Essex.

Seventy-six presents were announced as having been received since the last meeting.

THE DISTANCE AND DIRECTION TO THE GRAVITATIONAL CENTRE OF THE GALAXY FROM THE MOTIONS OF THE O5 TO B7 STARS.

J. S. Plaskett and J. A. Pearce.

Introduction

The theory of the rotation of the galactic system, developed by Lindblad and Oort, convincingly confirmed by observations at Victoria, has not only given dynamical unity to our conceptions of the galaxy and explained many hitherto puzzling motions of the stars, but has also provided an entirely independent method of obtaining the distance to the centre and, hence, the dimensions of the galactic system. These dimensions have hitherto mainly

been based on the distances of the globular clusters and these in turn primarily on the intrinsic luminosity of the Cepheids, which are present in some of the clusters. The first determination of Shapley, who developed the method, gave the distance from the Sun to the centre of the globular cluster system and, hence, to the centre of the galaxy as about 30,000 parsecs and the total diameter about 90,000 parsecs. These dimensions were later decreased by redetermination of the luminosity of the Cepheids, the zero point of the period-luminosity relation, to about 20,000 and 60,000 parsecs, but even these lessened dimensions met with the difficulty that the galaxy was still about five times larger in diameter than the largest known external system, the Andromeda nebula.

The gradual realization that interstellar space is not perfectly transparent, that there must be absorbing matter in the galaxy, again affected our estimates of the dimensions of the galaxy, as, in distances based on luminosities, absorption will make a star appear fainter and its distance will be estimated as too great. Various indirect attempts to determine the absorption gave widely differing values, and it was not until recent photo-electric measures of the colour excess of the globular clusters and of the B-type stars by Stebbins at Mt. Wilson, that any definite estimate of the effect of absorption was reached. Stebbins derived a distance of 10,000 parsecs from the Sun to the centre of the galaxy and a diameter of 30,000 parsecs, just half the previous estimate, but still leaving it two and a half times larger than the Andromeda.

There still remain some uncertainties in this method, and a determination of the dimensions by an independent method is obviously desirable. As indicated above, the distance to the gravitational centre as determined dynamically from the constants of the galactic rotation provides such a method, entirely independent of estimates of luminosity or space absorption. Previous determinations by this method of the distance to the centre by Oort, Lindblad and others gave values varying from about 6000 to 9000 parsecs, but the observational material was heterogeneous and a new determination from more homogeneous material and from stars at greater distances from the Sun is desirable.

Such material is provided by the stars of spectral classes O to B7, the most luminous stars in the sky, and, hence, the most distant that can be adequately spectroscopically observed. They have been under investigation at Victoria for ten years, and this paper presents a summary of some of the more important results derived from an analysis of their radial velocities and proper motions. A full discussion of the data will appear as the final number, No. 4, of volume v of the *Publications of the Dominion Astrophysical Observatory*, now in preparation. It will, however, be several months before this number will be published, and it has been thought desirable to give here some of the main results of the investigations.

The constants A and B of the galactic rotation are required, along with the orbital velocity, V, near the Sun, to determine the distance to the centre. The constants A and B and the direction to the centre can be obtained from an analysis of the proper motions alone, but values of the constant A and

the direction to the centre, l_0, can be obtained with much greater precision from an analysis of the radial velocities, leaving to the more uncertain proper motions, after assuming the radial velocity values of A and l_0, only the evaluation of the constant B.

The first section of this paper deals with the results of the analysis of the radial velocities of 849 O to B7 stars for the solar motion, K term, and galactic rotation. From the rotational terms are obtained l_0, the direction to the centre, and $\bar{r}A$, the rotational term, for several groups of stars at different distances. The values of A follow as soon as $\bar{r}$, the mean distance of the various groups are derived from the mean parallaxes. These in turn are obtained from the proper motions discussed in the second section of the paper. However, before the proper motions can be used safely to derive the rotational constant B, all known systematic motions, such as Raymond's correction to μ_δ, the appropriate solar motion, and the component of the southern stream motion, must be removed. There still remains the precessional corrections which should be derived anew from μ_b, the component of the proper motions in galactic latitude.

The value of the constant, B, can now be derived, after the application of the new precessional corrections to the already corrected values, from μ_l, the proper motion components in galactic longitude, after assuming in the equation the radial velocity values of A and l_0. The orbital velocity, V, is taken as 275 km./sec. from Strömberg's work, and the simple formula $V/R = A - B$ gives a distance to the centre of 10,000 parsecs, agreeing with Stebbins's value. The period of rotation in the neighbourhood of the Sun is about 220,000,000 years, and the total mass of the galactic system about 16×10^{10}. While the disparity in dimensions between the galaxy and the previously determined size of the Andromeda nebula of two and a half to one still remains, recent work by Hubble on globular clusters associated with the Andromeda, and photo-electric measures by Stebbins of the luminosity beyond the visible extension, both indicate that the Andromeda nebula has twice the previously accepted diameter or about 30,000 parsecs. The disparity in size now practically disappears and there is no longer need to consider our galactic system as a heterogeneous aggregation of discrete nebulæ, but it may now safely be considered as a large spiral nebula revolving in its own plane similar to the Andromeda and other external systems.

Analysis of the Radial Velocities

Observational Material.—The radial velocities used in this discussion have been taken from the catalogue, prepared by the authors,* consisting of the radial velocities of 1010 stars of spectral classes between O5 and B7, and from the list given by Neubauer † of the radial velocities of 213 southern B-type stars. Obviously for satisfactory analyses, besides the 146 B8 and B9 stars in the two lists which are outside the spectral range chosen, all uncertain and abnormally high velocities must be excluded. Consequently, 192 stars whose radial velocities had a probable error greater than $\pm 5{\cdot}3$

* *Pub. D.A.O.*, 5, 99, 1931. † *L.O.B.*, 15, 46, 1930.

km./sec. (only 13 stars between ± 5·0 and ± 5·3) were rejected. Similarly, 36 stars with velocities greater than about five times the random motion or 45 km./sec. were also rejected. There remained 260 O5 to B5 and 589 B3 to B7 radial velocities for analysis.

Preliminary Solutions.—If the radial velocities were to be analysed to determine the solar motion and K term only they could be used as they stand without arrangement into type or distance-groups. The introduction of terms in the solution to take care of the differential galactic rotation, whose magnitude depends on the distance of the stars, requires, however, that the stars be arranged into groups according to distance. The conditional equations should therefore be so formed as to enable a common solar motion and K term to be obtained from the radial velocities of all the 849 stars, and at the same time provide for a sufficient number of rotational terms so that no single group of stars covers too great a range in distance. The stars range in distance from about 100 to over 2000 parsecs with rotational terms, therefore, from about 2 to 30 km./sec. While it is desirable to have a small range of $\bar{r}A$ and distance in any one group, it is also necessary to have a sufficient number of stars in the group to prevent the disturbing effects of random motions from distorting the resultant values of the rotation.

The number of distance-groups, into which the stars should be divided for the optimum results, is therefore a compromise between the above conflicting requirements. As the labour in the formation of the normal equations and in the solutions in such a combined solution is very considerable, it seemed to us desirable to make some preliminary solutions for the galactic rotation alone, assuming the normal or standard solar motion of $V_0 = 20$ km./sec. towards the apex $\alpha = 271°$, $\delta = +28°$ or $l = 21°{\cdot}8$, $b = +20°$ in order to obtain some information leading to the most suitable division into distance-groups. The simple equation of condition for such solutions is obviously :

$$\rho = K + \bar{r}A \sin 2(l - l_0), \qquad (1)$$

an equation of three unknowns only and hence not entailing much labour.

A provisional division of the material into 7 distance-groups was first tested, the 260 stars containing interstellar lines being arranged into 2 groups as before,* of K intensities less than 5·0 and greater than 4·9 respectively. The 589 remaining stars were arranged into 5 groups according to spectral type and apparent magnitude as indicated in Table I below. Each of the 7 groups was divided into a suitable number of longitude subgroups, ranging from 12 to 17, fourth column of Table I, and solved from the above equation of condition by least squares with the results of Table I. The second column gives the spectral divisions whether O to B2 or B3 to B7 and the limiting magnitudes of the distance-groups, the third and fourth columns the number of stars and of longitude subgroups, while the other three columns contain the values of the unknowns with their probable errors, the upper value corresponding to this first solution.

An inspection of the values of $\bar{r}A$ and of l_0 indicates that this arrangement is not satisfactory, as particularly in the first two groups the values of $\bar{r}A$ are

* *Pub. D.A.O.*, 5, 190, 1933.

TABLE I

Preliminary Solutions

Group	Limits of Distance-Groups	Stars	Long. Groups	K	$\bar{r}A$	l_0
						° °
1	O–B2 < $3^m.5$: B3–B7 < $4^m.5$	108	13	+5·9 ± 0·8	1·3 ± 1·1	251 ± 35
				+2·5 ± 0·6	1·9 ± 1·0	368 ± 17
2	O–B2 $3^m.5$–$4^m.5$: B3–B7 $4^m.5$–$5^m.5$	184	17	+4·0 ± 0·6	2·4 ± 1·1	390 ± 9
				+1·8 ± 0·5	1·9 ± 0·5	355 ± 12
3	O–B2 > $4^m.5$	62	14	−3·7 ± 2·5	15·2 ± 3·2	311 ± 7
				−4·5 ± 2·5	16·4 ± 3·3	312 ± 7
4	B3–B7 $5^m.5$–$6^m.5$	115	14	+0·9 ± 0·3	2·0 ± 0·4	305 ± 6
				−1·1 ± 1·4	4·1 ± 0·7	311 ± 5
5	B3–B7 > $6^m.5$	120	12	−1·5 ± 0·9	3·7 ± 1·6	335 ± 8
				−1·7 ± 0·8	4·1 ± 1·0	335 ± 7
6	Ca^+ Intensity < 5·0	158	15	+4·3 ± 1·0	7·8 ± 1·5	309 ± 6
				+4·2 ± 1·0	7·8 ± 1·4	309 ± 6
7	Ca^+ Intensity > 4·9	102	12	−0·2 ± 1·8	19·8 ± 2·6	330 ± 4
				−0·2 ± 1·8	19·8 ± 2·6	330 ± 4

so uncertain and of l_0 so far removed from the geometrical centre at 327°, that they contradict rather than confirm normal rotational movement. It is obvious that either the numbers of stars in these two groups are insufficient or the distances are too small to give definite results. A further difficulty, not at first realized is that the average absolute magnitudes of the O to B2 stars are two magnitudes brighter than those of the B3 to B7 stars, instead of the one magnitude assumed in the arrangement, and it seems obvious that the two spectral divisions should not be grouped together. Again, when the majority of the stars are arranged according to spectral type and apparent magnitude, it seems inconsistent to arrange the remaining 30 per cent. according to the intensity of the interstellar lines. These preliminary solutions therefore seemed to indicate that: (*a*) The numbers of stars in a group, particularly among the nearer stars, should be increased, thus decreasing the number of groups; (*b*) the O to B2 stars being so much more distant should not be grouped with the B3 to B7 stars; (*c*) the basis of grouping should be uniform, depending on type and apparent magnitude only.

Stream Motion among the Southern Stars.—The velocities computed from the values of the unknowns, especially in the first four groups, when compared with the observed velocities, showed a marked systematic difference in the longitude subgroups in the southern hemisphere in the sense that the observed velocities were markedly more positive than the computed. A similar effect was noted in some earlier solutions of the motions of the B stars,* but it was not then identified with the strong community of motion among the southern stars between galactic longitudes 216° and 360° investigated by Kapteyn.† The reappearance of this systematic outward motion in these solutions led to a comparison with Kapteyn's results, and this unmistakably

* *Science*, **71**, 225, 1930; *M.N.*, **90**, 630, 1930. † *Ap. J.*, **40**, 43, 1914.

pointed to the identity of the two effects and led to a redetermination of the stream velocity, the convergent point, and the radial component of the stream. The redetermination was necessary as the number of radial velocities available was not only doubled and the accuracy increased, but also because the application of Raymond's correction to the proper motions in declination would undoubtedly change the position of the convergent.

The precessional corrections suggested by Boss were first of all applied to the tabular proper motions of the O to B7 stars between galactic longitudes 210° and 360° and latitudes ± 30° and then Raymond's correction in μ_δ. The corrected μ_α, μ_δ, μ with its probable error, and the position angle ψ were then derived and tabulated. All proper motions less than 0″·015 per annum and all with probable errors greater than ± 0″·015 were rejected in order to avoid including uncertain members of the stream. Also any stars whose motion deviated from the general trend by 90° or more were also excluded. There remained about 200 possible members, and these were arranged into 8 groups with limits as indicated in the third column of Table II. These groups deviate somewhat from those chosen by Kapteyn, but appear to suit better the positions and motions of the component stars and to arrange them into more nearly equal groups. A preliminary determination of the mean position angle ψ of each group, then allowed a more critical determination of the stream members. Among the 154 stars assigned as probable members, the maximum deviation from the mean value of the group allowed was 70°. Actually there were only 4 stars with deviations between 50° and 70°, and 17 with deviations between 30° and 50°, while the average deviation was only 15°.

TABLE II

Data of Southern Stream

Group	Limits Galactic Co-ordinates	Proper Motions					Radial Velocities			
		No.	$\bar{\mu}$	$\bar{\psi}$	$\delta\psi$	$\bar{D}$	No.	$\bar{V}$	$\bar{K}$	K'
	° ° ° °		″	°	°	°				
1	210–240 : −15 to + 8	25	·0253	291·2	+0·8	142·6	20	+17·8	+ 5·9	+5·9
2	240–260 : −31 to +20	19	·0229	288·0	+1·8	126·9	18	+18·3	+ 9·3	+7·7
3	260–280 : −20 to + 2	17	·0361	249·4	−1·7	106·8	15	+12·9	+ 8·5	+9·2
4	260–280 : + 2 to +24	19	·0376	240·8	−3·1	99·5	15	+14·4	+11·9	+9·5
5	280–305 : −20 to +10	15	·0291	209·8	+2·4	81·4	13	+ 6·3	+ 8·5	+9·6
6	280–305 : +10 to +30	24	·0372	221·1	−1·7	77·3	23	+ 5·7	+ 9·0	+9·4
7	305–330 : +10 to +30	24	·0308	206·9	−0·5	53·3	18	− 2·5	+ 6·5	+7·8
8	305–330 : −18 to +10	11	·0357	185·0	+2·1	62·8	10	+ 1·0	+ 7·8	+8·6

The convergent point of the stream was easily determined by the aid of a very accurately turned and ruled metal globe with an accessory graduated metal quadrant with attached pole. From the centres of gravity of the groups, great circles in position angle $\bar{\psi}$ were drawn, and the convergent point was the mean of the intersections of these lines, weighted proportion-

ally to the included angle. While it could hardly be expected, with such small and relatively uncertain proper motions, that these great circles should intersect in a single point, the deviations from the mean convergent $\delta\psi$, sixth column Table II, are so small as to give confidence in the reality of the stream. The position of the convergent is

$$\alpha = 83^\circ\cdot6 : \qquad \delta = -22^\circ\cdot9,$$
$$l = 194^\circ\cdot2 : \qquad b = -24^\circ\cdot3.$$

Kapteyn obtained two convergents, the first directly from the proper motions and the second after a provisional correction in declination :

$$1.\ \alpha = 94^\circ\cdot5, \qquad \delta = -42^\circ,$$
$$2.\ \alpha = 96^\circ\cdot0, \qquad \delta = -39^\circ.$$

The distance between the convergent obtained above and Kapteyn's first is $21^\circ\cdot0$ and second $19^\circ\cdot6$, while the distance from the solar vertex $\alpha = 91^\circ$: $\delta = -28^\circ$ is only $8^\circ\cdot5$, and hence the stream motion differs little in direction from the reflex of the solar motion.

The simple equation for determining the stream motion is

$$V_0 \cos \bar{D} + K = \bar{V}, \tag{2}$$

where V_0 is the stream velocity, $\bar{D}$ the angular distance from the vertex $\alpha = 263^\circ\cdot6$: $\delta = 22^\circ\cdot9$ given in column seven, K the average outward radial component of the stream, and $\bar{V}$ the mean radial velocity. While there are 154 members of the stream only 132 of these stars have usable radial velocities, the number and the mean velocity in each group being given in columns eight and nine. A solution for V_0 and K of these 8 groups gives the following values :—

Stream velocity $V_0 = -15\cdot00 \pm 1\cdot04$ km./sec.
Average radial component $K = +\ 8\cdot33 \pm 0\cdot50$ km./sec.

These values of V_0 and K allow another determination of the convergent point to be obtained, as its distance along the direction line from the centre of gravity of each group can be obtained from the above equation transposed thus :

$$\cos \bar{D} = \frac{\bar{V} - K}{V_0}. \tag{3}$$

The weighted mean value from the eight separate determinations is

$$\alpha = 84^\circ\cdot4 : \qquad \delta = -23^\circ\cdot7,$$

which is only $1^\circ\cdot0$ from the previous value, a very satisfactory agreement considering the relative uncertainty of the observational data. A determination of V_0 and K from this convergent gave

$$V_0 = -15\cdot03 \pm 0\cdot95 \text{ km./sec.}$$
$$K = +\ 8\cdot11 \pm 0\cdot47 \text{ km./sec.}$$

which are practically identical with the previous values. The earlier values were, however, adopted in subsequent analysis.

The value of the radial component of the stream of +8·3 km./sec. is the average over the whole group, but, owing to the considerable extent of the group, of 120° in longitude, the radial component will be larger at the middle and smaller at the ends of the group. The observed values are obviously given by the equation

$$K = \overline{V} - V_0 \cos \overline{D} \qquad (4)$$

and are shown in the next to the last column of Table II. These are affected by the accidental errors, and it seemed preferable to use the smoothed values of K which will obviously be proportional to sin D and which must be adjusted so that the average K for the whole stream is +8·33 km./sec. These smoothed values are given in the last column of the table and are the velocities which should be subtracted from the velocities of the members of the corresponding groups to free them from the effects of the stream motion.

The question naturally arises whether other stream or cluster motions among these stars should be considered. The best known are in the constellations of Orion and Monoceros, the Pleiades, and the h and χ clusters in Perseus. These are, however, quite limited in area, include few stars, while generally speaking the stream motion is uncertainly determined. In addition the streaming, wherever suspected in these areas, is so similar to the component of the galactic rotation, for the corresponding direction and distance, that any residual effect may safely be considered accidental. Hence, no correction for systematic streaming was applied except in the southern sky.

The effect of making this correction to the observed radial velocities in the preliminary solutions is shown in the second set of values of the unknowns in Table I. An inspection of these columns shows unmistakably the improvement in the run of the values, particularly in the first two groups of the nearer stars where the values of l_0, though still abnormal, are much better than the uncorrected ones. The probable errors of the determinations of $\bar{r}A$ and l_0 are also in general materially reduced, and there can be no doubt that the removal of this systematic effect has improved the determinations. The weighted mean value of l_0 as obtained from the two determinations are from the observed velocities $l_0 = 324°{\cdot}5 \pm 6°{\cdot}1$, and when southern velocities are corrected $l_0 = 322°{\cdot}8 \pm 3°{\cdot}8$.

The General Solution.—The results of the preliminary solutions have clearly indicated, first, the necessity of removing the radial component of the southern stream from the participating stars, and this was done first of all; and, secondly, the general principles to be followed in arranging the stars into distance-groups before making the combined solution. It seemed evident that the numbers of stars, in the nearer groups particularly, should be increased, that the O to B2 stars should not be combined with the B3 to B7, and that a single criterion only, that of spectral class and apparent magnitude, should be used in the arrangement. Consequently, the number of distance-groups were limited to four, brighter and fainter O to B2 stars and brighter and fainter B3 to B7 stars. For the latter, as there is little

range in absolute magnitude in the spectral subdivisions in the group, the stars were simply divided into two groups, with 272 stars brighter than $5^{m}\cdot50$ and with 317 stars fainter than $5^{m}\cdot49$. For the O to B2 stars, however, as will be seen from the mean parallaxes later, those of Class O are nearly one magnitude brighter than those of Classes Bo to B2. Consequently, the magnitude division was placed at $5^{m}\cdot0$ for Class O and $6^{m}\cdot0$ for Bo to B2, there being 101 stars in the brighter and 159 in the fainter group. The arrangement is repeated in Table III.

The complete equation of condition from which the systematic terms, the K term, the solar motion and the galactic rotation may be derived is well known :

$$V = K + X \cos l \cos b + Y \sin l \cos b + Z \sin b + \bar{r}A \sin 2(l - l_0) \cos^2 b, \qquad (5)$$

where X, Y, Z are the rectangular components of the solar motion, l, b the galactic co-ordinates of the star, $\bar{r}A$ is the rotational effect and l_0 the galactic longitude of the centre of rotation. Owing to the concentration of the stars to the galactic plane, the latitude of the solar apex, B_0, is practically indeterminate. The value of $+20°$ was assumed for B_0 and only the projected solar velocity $V_0 \cos B_0$ determined. Equation (5) then reduces to

$$V = K + X \cos l + Y \sin l + \bar{r}A \sin 2(l - l_0) \cos^2 b, \qquad (6)$$

which, as is evident from what precedes, is only properly applicable to a group of stars of small range in distance. The 849 stars have, however, just been divided into four groups at different mean distances, and equation (6) must be expanded to obtain the four different values of $\bar{r}A$ and l_0, but only single values of the solar motion and K term. This obviously becomes

$$V = K + X \cos l + Y \sin l + \bar{r}_1 A \sin 2(l - l_1) \cos^2 b + \bar{r}_2 A \sin 2(l - l_2) \cos^2 b + \bar{r}_3 A \sin 2(l - l_3) \cos^2 b + \bar{r}_4 A \sin 2(l - l_4) \cos^2 b, \qquad (7)$$

which becomes, when the rotational terms are expanded,

$$V = K + aX + bY + c_1 x_1 + d_1 y_1 + c_2 x_2 + d_2 y_2 + c_3 x_3 + d_3 y_3 + c_4 x_4 + d_4 y_4, \qquad (8)$$

where

$$a = \cos l, \qquad b = \sin l,$$
$$c_1 = \sin 2l \cos^2 b, \qquad d_1 = \cos 2l \cos^2 b, \text{ etc.},$$
$$x_1 = \bar{r}_1 A \cos 2l_1, \qquad y_1 = -\bar{r}_1 A \sin 2l_1, \text{ etc.},$$

an equation of eleven unknowns.

In forming the normal equations for the four distance-groups, observation equations were not formed for each star, individually, as in the solutions for the interstellar motions,* but they were arranged into a number of longitude subgroups, 67 in all, and the mean velocity of each subgroup was taken as the absolute term in the observation equation for that group. The principal advantage of taking individual stars was retained, however, by forming the coefficients of the unknowns, $\cos l$, $\sin l$, $\sin 2l \cos^2 b$,

* *Pub. D.A.O.*, 5, 189, 1933.

$\cos 2l \cos^2 b$, separately for each star and taking the means for each subgroup instead of taking the means of l and b and then forming the coefficients. The differences by the two methods are considerable in some of the longitude subgroups, and there is no doubt that the former method is preferable, and will give practically the same results as if an observation equation were formed for each star and with less than a tenth of the labour. The normals of the four distance-groups were formed separately and then combined to make the normal for the complete general solution.

The solutions of the solar motion, K term and galactic rotation, normals of five unknowns for the four groups, were carried through separately and were followed by the solution of the combined normal equations with eleven unknowns. This solution, though tedious, is not difficult, and followed the usual methods. It is not deemed necessary to give the values of the unknowns as solved, but only the values of the systematic motions derived from them. These are given in Table III in the second column for the

TABLE III

Values of the Systematic Motions

	General Solution	Group 1—101** O5–O9 < 5·0 Bo–B2 < 6·0	Group 2—159** O5–O9 > 4·99 Bo–B2 > 5·99	Group 3—272** B3–B7 < 5·50	Group 4—317** B3–B7 > 5·49
K	+ 1·13 ± 0·40	+ 2·41 ± 0·66	+ 1·87 ± 2·15	+ 2·68 ± 0·33	− 0·90 ± 0·76
$V_0 \cos B_0$	17·94	16·46	19·35	19·79	16·12
V_0	19·09 ± 0·57	17·51 ± 1·05	20·60 ± 1·11	21·06 ± 0·49	17·16 ± 0·91
L_0	21°·4 ± 1°·8	1°·2 ± 3°·4	31°·0 ± 12°·4	19°·9 ± 1°·3	19°·1 ± 4°·2
$\bar{r}_1 A$	+ 9·74 ± 1·59	+ 9·80 ± 1·20	...		
l_1	326°·7 ± 5°·1	325°·9 ± 3°·0	...		
$\bar{r}_2 A$	+18·00 ± 1·24	...	+16·73 ± 2·61		
l_2	322°·3 ± 2°·2	...	324°·1 ± 4°·3		
$\bar{r}_3 A$	+3·44 ± 1·33	...	...	+ 3·30 ± 0·67	
l_3	334°·2 ± 4°·0	...	...	331°·9 ± 3°·2	
$\bar{r}_4 A$	+ 3·33 ± 0·86	...	...	...	+ 4·46 ± 0·95
l_4	308°·0 ± 8°·1	...	...	...	308°·9 ± 7°·1

general or combined solution, and in the four succeeding columns are the individual solutions for the four distance-groups. It is interesting to note the scatter in the individual values of the solar motion and K term for the four groups, and also to find that the mean values, weighted according to the number of stars, are

$$K = +1{\cdot}16 \pm 0{\cdot}63 \text{ km./sec.}; \quad V_0 = 19{\cdot}10 \pm 0{\cdot}64 \text{ km./sec.}; \quad L_0 = 19°{\cdot}46 \pm 3°{\cdot}14,$$

practically identical, as perhaps might be expected, with the values of the combined solution. There are also some curious differences in the rotational terms between the combined and individual solutions which will be later considered.

The Solar Motion.—It is of interest to compare the solar velocity and solar apex derived from the radial velocities of 849 O to B7 stars with other determinations from similar spectral classes and with those from all classes together. The principal values are summarized in Table IV, the comparison of determinations from stars of Class B being given in the first part and from stars of all classes in the second part of the table. On account of the galactic concentration of the Class B stars, the solar apex is difficult to determine and has frequently been assumed as $A_0 = 270°$, $D_0 = +30°$, corresponding to $L_0 = 23°{\cdot}5$, $B_0 = +21°{\cdot}6$, as shown by the values in parentheses in Table IV. The determination from the most accurate material is that of Pearce from the radial velocities of 680 O to B5 stars with probable errors less than ± 3·0 km./sec. In this analysis terms were included for galactic rotation, the resulting longitude of the centre l_0 being $321°{\cdot}5 \pm 1°{\cdot}9$ and $\bar{r}A = +4{\cdot}3 \pm 0{\cdot}8$ km./sec. Similar solutions for the apex, from two subdivisions in spectral type and two in magnitude, showed that the latitude of the apex for the O to B2 stars, for example, which are highly concentrated to the galaxy, is practically indeterminate. This led us, in subsequent work, to adopt $B_0 = +20°$, about midway between solutions (11) and (14) and the value adopted for the graphs of Pearce and Hill,* and solve for the projected solar motion.

TABLE IV

Various Values of the Solar Motion

No.	Solutions	No.	Types	K	V_0	L_0	B_0
						°	°
1	Gyllenberg, K and Solar Motion only	284	B	+4·3	22·1	25·4	+ 6·7
2	Wilson, ,, ,, ,, ,,	384	Bo–B5	+3·5	20·2	(23·5)	(+21·6)
3	Frost, Barrett, Struve, Solar Motion only	350	O–B9	+5·6	17·2	12·1	+ 2·2
4	Campbell, Moore, K and Solar Motion only	284	O–B5	+4·9	22·7	(23·5)	(+21·6)
5	P and Pe, K, Solar Motion, Gal. Rot.	314	Interstr.	...	21·1	22·0	(+20·0)
6	Pearce, ,, ,, ,, ,, ,,	680	O–B5	+3·5	22·0	36·8	+17·3
7	P and Pe, ,, ,, ,, only	849	O–B7	+3·1	21·7	32·1	(+20·0)
8	P and Pe, K, Sol. Mot., Gal. Rot., Stream Mot.	849	O–B7	+1·1	19·1	21·4	(+20·0)
9	Charlier, Solar Motion from Rad. Vel.	1986	B–M	...	19·0	22·9	+23·8
10	,, ,, ,, ,, Proper Mot.	4182	O–M	...	(19·0)	26·1	+21·4
11	Wilson, ,, ,, ,, Rad. Vel.	2748	,,	...	19·0	21·0	+19·8
12	,, ,, ,, ,, Proper Mot.	2305	,,	...	...	20·8	+19·8
13	,, K, ,, ,, ,, Rad. Vel.	3761	,,	+0·4	19·6	(23·5)	(+21·6)
14	Campbell and Moore, Sol. Mot. from Rad. Vel.	2149	,,	+1·2	19·6	23·9	+21·2

Note.—Values in brackets are assumed values.

It will be observed that the solutions 1–7 of Table IV give values for the K term varying from +3·1 to +5·6 km./sec., and a value of the solar velocity, of the order of 22 km./sec., some 3 km./sec. higher than the values derived from the stars of all classes, solutions 9–14. That the K term and this high

* *Pub. D.A.O.*, 4, 49, 1927.

solar velocity are probably due to the systematic recession of the bright B stars in the southern sky is evident from the results of solution 8. Where this stream motion is eliminated, the K term is reduced to $+1{\cdot}1$ km./sec., and the projected solar velocity is identical within errors of observations with that determined from the naked-eye stars. That the solar motion is similar to that derived from the solutions for stars of all spectral types is satisfactory confirmation of the correctness of this procedure, and indicates that the vertex and velocity of the southern stream of Class B stars has been correctly determined and eliminated.

The K *Term.*—The high positive value of the average residual velocity, the K term, of some 4 or 5 km./sec., obtained from earlier solutions of the radial velocities of the B-type stars, has always been a puzzling phenomenon, and various attempts by von der Pahlen and Freundlich,* by Rosenhagen † and others, have not succeeded in solving the cause of this apparent expansion of the system of B stars. In earlier preliminary work ‡ on these stars it was shown to be much reduced among the northern stars by the introduction of terms for the galactic rotation, while the assumption of a systematic outward trend in the velocities of the southern stars removed the difficulty in that region of the sky. Only now, however, has this outward motion been identified with the radial component of the southern stream, earlier investigated by Kapteyn, and the proper correction determined and applied to the radial velocities of the members of the stream.

The K term has been reduced by this correction and by allowing for the galactic rotation to $+1{\cdot}1 \pm 0{\cdot}4$ km./sec., from $+4{\cdot}9$ for the Lick solutions, $+5{\cdot}6$ for the Yerkes, and from $+3{\cdot}5 \pm 0{\cdot}5$ and $+3{\cdot}1 \pm 0{\cdot}6$ km./sec. from the incompletely corrected Victoria velocities, as seen in Table IV. While previous attempts have been made to explain the K term as possibly due to the Einstein gravitational displacement of the spectral lines of massive stars, these required the assumption of abnormally high masses or densities to give a displacement of 5 km./sec. For example, for a star of mass 10 times the Sun, an average mass for a B-type star, the density would require to be nearly 6 times the Sun to produce an Einstein displacement of 5 km./sec. Similarly for a density of 0·07 times the Sun, about that to be expected for an average B-type star,§ a mass of 80 times the Sun would be required, many times the average mass. It is obviously quite impossible to explain a K term of 5 km./sec. by the gravitational shift, but it is another matter when reduced to $+1{\cdot}1$ km./sec.

In order to compute the probable value of the Einstein shift for these stars, it is necessary to have average values of the masses and densities, the displacement being directly proportional to the mass and inversely to the radius. The most recent values known to us are those compiled by Plaskett,‖ of which the significant values are repeated in Table V, with the corresponding numbers of stars of the three spectral divisions used in the solution. The

* *Publ. A.O. Potsdam*, **26**, No. 3, 1928.
† *A.N.*, **242**, 401, 1931.
‡ *Science*, **71**, 225, 1930; *M.N.*, **90**, 631, 1930.
§ *M.N.*, **90**, 621, 1930. ‖ *Ibid.*

TABLE V

Average Values of Mass, Density and Gravitational Shift

Spectral Divisions	No.	Mass	Density	Red-shift
O	63	37·8	0·049	+2·62
Bo–B2	197	18·1	0·056	+1·67
B3–B5	589	7·4	0·071	+1·00

computed values of the velocity displacements for the three divisions are given in the last column, and the mean, weighted according to the number of stars, is +1·27 km./sec., a value practically the same, within the probable errors, as the value of +1·13 ± 0·40 km./sec. obtained in the general solution. The famous *K* term in the Class B stars, whose high positive value of nearly +5 km./sec. has hitherto successfully evaded explanation, has now been reduced, by taking into account all known systematic motions, to +1·1 km./sec., and this remaining displacement is practically the same as the computed Einstein red-shift of the spectral lines in these stars.

The Galactic Rotation.—The preliminary solutions in which the standard solar motion was assumed were admittedly only provisional, and while the rotational terms were approximately of the expected magnitude, the directions to the centre, especially for the two groups of nearer stars, were widely discrepant. As the method of arrangement into distance-groups was changed and a combined solution made, it is these latter values only, which are given in the second column of Table III, that will be used in later work. The values of $\bar{r}A$ for the four groups are all of the right order, although the mean parallaxes which are discussed later in the paper indicate that $\bar{r}A$ for the first group of brighter O to B2 stars is about 30 per cent. larger, and for the fourth group of fainter B3 to B7 stars about 50 per cent. smaller, than would be expected from the mean parallaxes. In the first group there are only 101 stars which extend over a considerable range in apparent magnitude between $1^{m}\cdot2$ and $6^{m}\cdot0$. It may well be that the group contains a disproportionate number of very luminous stars, super-giants, more distant than indicated by the basis of division, apparent magnitude, and hence tending to give a high value of $\bar{r}A$. In the fourth group of the fainter B3 to B7 stars there are ample numbers, 317, and there is no obvious explanation of why the value of $\bar{r}A$, obtained either by the combined or individual solutions, should be only about half the value expected from the mean parallax. There is apparently some unexplained systematic motion present in this group of stars which not only reduces the value of $\bar{r}A$ but also displaces the direction to the centre by nearly 20° from the normal position. This possibility is discussed in the complete publication, but it cannot well be touched upon in the limited space available here.

Just as in earlier discussions of the radial velocities of distant stars, and also of the interstellar matter, the evidence for the existence of a differential

rotational effect in the motions of these stars is very direct and convincing. This is shown by the relatively small probable errors, by the agreement of the directions to the centre with one another and with the geometrical centre of the globular cluster system, and also by a comparison of the observed velocities with those computed from the values of the unknowns in the solutions. Table VI contains a representation of the observed velocities, corrected for stream motion, with the velocities computed from the values of the combined solution for groups 3 and 2, the nearest and most distant of the groups. The third column of each section of the table contains the mean longitude, the fourth the mean observed velocity, corrected for the southern stream, and the fifth the computed velocity, whose close agreement is confirmation of the completeness with which the solar motion and the galactic rotation represent the observed velocities. Any deviations may in general

TABLE VI

Comparison of Observed and Computed Velocities

Group 3—Brighter B3–B7 Stars					Group 2—Fainter O–B2 Stars				
Group	No.	l	$\overline{V}$	V'	Group	No.	l	$\overline{V}$	V'
		°					°		
1	6	2·4	−14·9	−14·5	1	4	1·9	+ 1·4	− 1·6
2	10	23·0	−14·2	−15·2	2	8	27·1	− 4·6	− 4·3
3	14	38·7	−13·0	−14·9	3	25	43·8	−16·8	−11·4
4	12	50·8	−12·0	−14·0	4	12	60·8	− 9·4	−18·6
5	9	67·3	−13·8	−12·6	5	17	71·5	−20·3	−22·0
6	9	89·7	− 9·3	− 8·0	6	9	89·0	−20·9	−23·0
7	13	115·7	+ 2·3	− 0·6	7	19	106·3	−25·1	−17·4
8	11	132·1	+ 9·5	+ 5·8	8	8	122·6	+ 2·7	− 6·0
9	14	149·2	+14·6	+12·3	9	9	144·7	+ 7·2	+ 1·9
10	13	165·2	+22·6	+17·6	10	8	166·4	+20·8	+28·5
11	13	182·5	+20·2	+21·6	11	20	175·3	+33·7	+34·0
12	18	205·1	+28·3	+23·0	12	5	191·3	+37·6	+37·6
13	16	218·3	+19·6	+21·8	13	1	231·3	+23·5	+17·9
14	16	235·2	+20·2	+17·9	14	2	277·7	− 9·1	−11·7
15	17	250·3	+14·8	+13·1	15	5	321·0	−16·9	− 9·0
16	20	267·8	+ 9·3	+ 6·4	16	6	337·5	+ 9·6	− 3·7
17	23	288·5	− 0·8	− 0·9					
18	15	302·1	− 4·0	− 5·7					
19	18	319·5	− 9·7	− 9·3					
20	5	342·2	−10·0	−12·9					

be safely ascribed to accidental errors, or abnormal peculiar motions in some of the stars in longitude groups with few stars. Fig. 1 shows graphically in heavy lines the final curves from the general solution, not only of these two but of the whole four groups. The mean velocities of the longitude groups corrected for the southern stream are plotted to indicate the fit given by the analysis. The curves in light lines are the single-wave solar motion and the

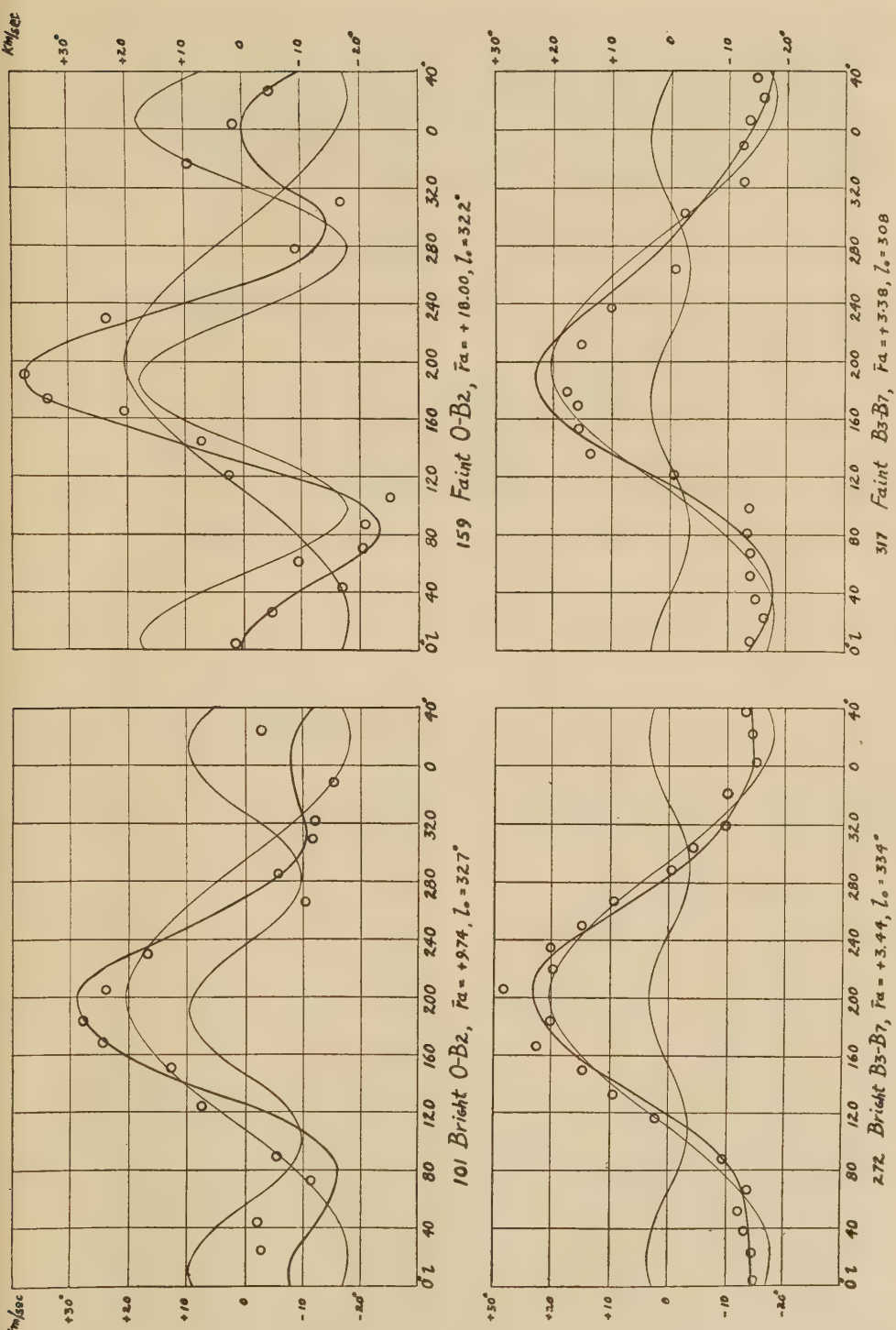


FIG. 1.—*Graphical Representation of Observations and of the Curves from the General Solution.*

double-wave galactic rotation which with the constant K term of $+1{\cdot}1$ km./sec. make up the heavy line final curves.

The Direction to the Centre.—The values of the direction to the centre of rotation for the four distance-groups are given in the second column of Table III for the combined solution and in the next four columns for the individual solutions. The weighted mean values are :

For the combined solution, $l_0 = 324^\circ{\cdot}4 \pm 2^\circ{\cdot}3$,
For the individual solutions, $l_0 = 326^\circ{\cdot}4 \pm 2^\circ{\cdot}2$.

While of low weight, the mean values of the preliminary solutions may be repeated here :

For the observed velocities, $l_0 = 324^\circ{\cdot}5 \pm 6^\circ{\cdot}1$,
For the corrected velocities, $l_0 = 322^\circ{\cdot}8 \pm 3^\circ{\cdot}8$.

Some of the principal values of l_0 are collected in Table VII. The weighted mean value of the rotational determinations is $325^\circ{\cdot}5 \pm 1^\circ{\cdot}1$, which

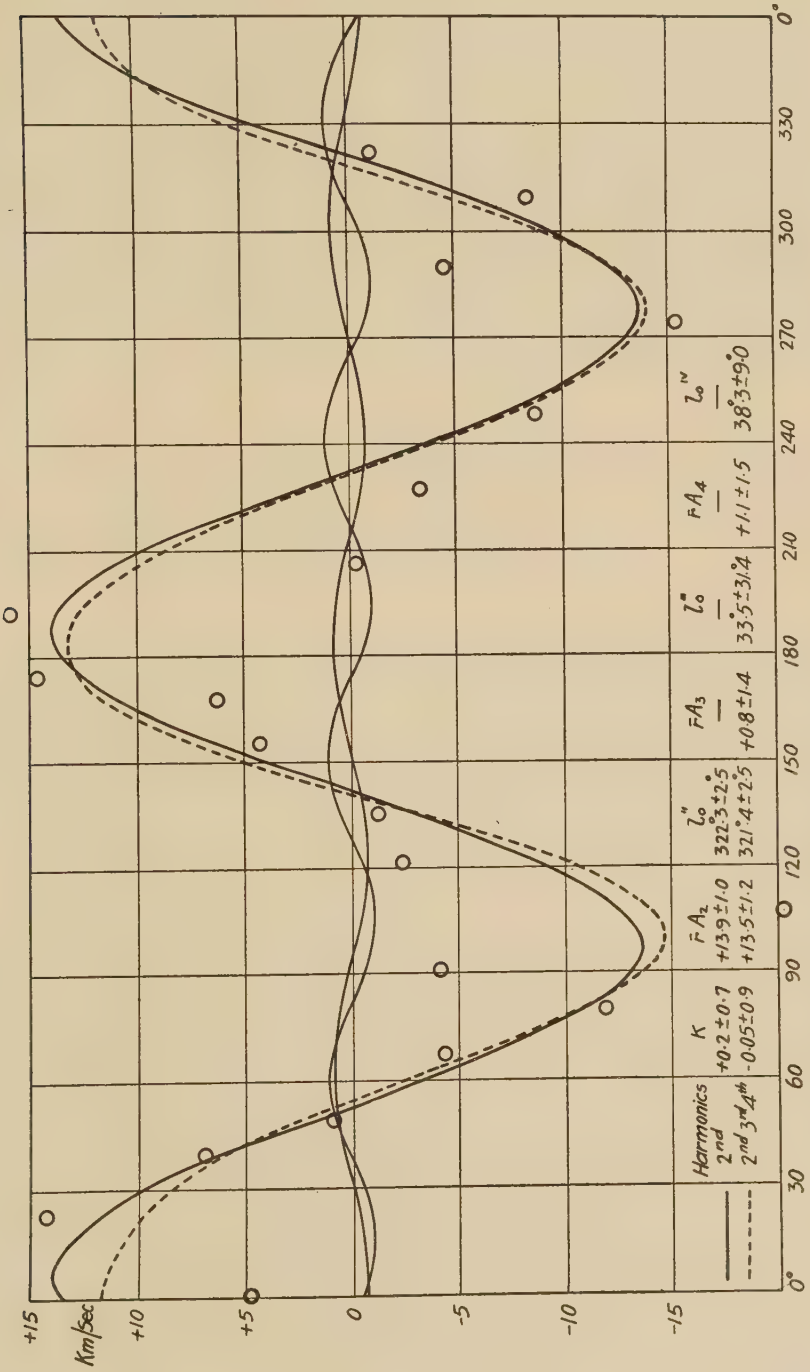
TABLE VII

Principal Values of l_0

Authors and Material	Apex	l_0
	° °	° °
Oort, Rad. Vel. of about 700 O–B5, A–G, "c" stars, Planetaries	270 + 30	324 ± 2
P and Pe, Rad. Vel. from Interstellar lines in 314 stars	271 + 28	331·1 ± 1·8
Pearce, ,, ,, of 680 O–B5 stars : p.e. < ± 3·0 km./sec.	281 + 40	321·5 ± 1·9
P and Pe, ,, ,, of 849 O–B7 stars : p.e. < ± 5·3 km./sec.	271 + 28	324·4 ± 2·4
Oort, Proper Mot.—771 O, B, "c," N, Cepheids	...	326 ± 11
Oort, ,, ,, 5413 Boss stars	...	333 ± 10
P and Pe, ,, ,, 717 O–B7 stars	...	311 ± 9
Shapley, Direction of Geometrical Centre of Globular Clusters	...	327 ± 2

agrees well with the geometrical value of 327°. The cause of the high value of l_0 for the interstellar matter is not evident, but the difference so much exceeds the probable error that it must be real and probably due to some peculiarity of distribution or velocity. The small value of l_0 in Pearce's solution may easily have arisen on account of the effect of the least squares when the practically indeterminate latitude is included in the solution. It was decided, however, to accept the value of $324^\circ{\cdot}4$ given by the general solution in subsequent work on the same stars.

Additional Harmonic Terms.—The simple rotational equation of Oort is based on the assumption that the distance r to the stars concerned is small compared to the distance R to the gravitational centre, so that higher order



$$\bar{p} = 0{\cdot}2 + 13{\cdot}9\,\mathrm{Sin}\,2(\bar{l} - 322{\cdot}3^\circ)\,\mathrm{Cos}^2\bar{b}$$
$$\bar{p} = 0{\cdot}05 + 13{\cdot}5\,\mathrm{Sin}\,2(\bar{l} - 321{\cdot}4^\circ)\,\mathrm{Cos}^2\bar{b} + 0{\cdot}8\,\mathrm{Sin}\,3(\bar{l} - 33{\cdot}5^\circ)\,\mathrm{Cos}^2\bar{b} + 1{\cdot}1\,\mathrm{Sin}\,4(\bar{l} - 38{\cdot}3^\circ)\,\mathrm{Cos}^2\bar{b}$$

FIG. 2.—*Representation of Residual Velocities of 252 O to B2 Stars with 2nd, 3rd and 4th Harmonic Terms.*

terms may be neglected. Miss Hayford,* in discussing the radial velocities of stars in the open clusters, used third and fourth harmonic terms in addition to the single-wave solar motion and double-wave galactic rotation, and obtained values of the third and fourth harmonics nearly half as large as the second rotational term. The radial velocities used were obtained from low dispersion spectrograms and were rather uncertain, and it seemed worth while to test the presence of higher harmonics by an analysis of the radial velocities of the stars of Classes O to B2, groups 1 and 2, the most distant available.

Equation (18) of Miss Hayford's paper, simplified by assuming the solar motion of the general solution, was employed. The O to B2 stars were rearranged into 27 longitude subgroups and a least-squares solution made, resulting in the following values :—

$$\begin{aligned} K &= -\ 0{\cdot}05 \pm 0{\cdot}89, \\ \bar{r}_2 A &= +13{\cdot}54 \pm 1{\cdot}20, \\ l_2 &= 321^\circ{\cdot}4 \pm 2^\circ{\cdot}5, \\ \bar{r}_3 A &= +0{\cdot}80 \pm 1{\cdot}38, \\ l_3 &= 38^\circ{\cdot}5 \pm 31^\circ{\cdot}4, \\ \bar{r}_4 A &= +1{\cdot}13 \pm 1{\cdot}46, \\ l_4 &= 38^\circ{\cdot}3 \pm 9^\circ{\cdot}0. \end{aligned}$$

It will at once be noted that the third and fourth harmonics are so insignificant in comparison to the second rotational term and to their accidental errors, that it may safely be said that r/R is so small that higher order terms may be neglected. The relative importance of third and fourth harmonics is well illustrated by the graph of fig. 2, which shows the negligible change produced in the rotational curve by their introduction, the heavy continuous curve giving the representation by the complete solution.

Analysis of the Proper Motions

Observational Material.—The proper motions of 535 of the stars in the two catalogues mentioned above were obtained from the Boss Catalogue. Through the kind collaboration of Dr. Benjamin Boss and Dr. R. E. Wilson, the proper motions of 361 additional O5 to B7 stars were provided in advance of publication. There were hence available, the proper motions of 896 stars of types between O5 and B7, but for 15 of these stars the proper motions were so abnormally large as to be unsuitable for statistical treatment. There remain 881 proper motions as compared with 849 radial velocities, but only 765 of these stars have both proper motions and radial velocities suitable for analysis. It was, however, deemed advisable to use the whole 881 stars with proper motions without reference to whether they have radial velocities. The probable errors of these proper motions are of widely differing precision, ranging from $\pm 0''{\cdot}002$ to $\pm 0''{\cdot}03$, and following the procedure † in the investigation of the interstellar matter, weights were assigned as follows : When the probable error is

* *L.O.B.*, 16, 53, 1932.

† *Pub. D.A.O.*, 5, 202, 1933.

Less than	± 0″·0050	the weight was		2·0,
From	0″·0050 to 0″·0099	,,	,,	,, 1·0,
From	0″·0100 to 0″·0149	,,	,,	,, 0·7,
From	0″·0150 to 0″·0199	,,	,,	,, 0·4,
Greater than	± 0″·0200	,,	,,	,, 0·2.

Mean Parallaxes and Absolute Magnitudes.—Raymond's revised correction to the proper motions in declination and the precessional corrections suggested by Boss were applied individually to these 881 proper motions, and they were then transformed into the v and τ components, parallel and perpendicular to the solar motion. While mean parallaxes were derived for comparison from the whole 881 proper motions, it was considered that more reliable values would be obtained by discarding the 42 proper motions with probable errors greater than ± 0″·0149 and using only the 839 fairly good proper motions.

It seemed desirable to divide the stars into as many type and magnitude groups as possible, consistent with retaining a sufficient number in each group to ensure reliable mean parallaxes. Subdivision into 13 groups, as indicated in Table VIII, was finally determined. The well-known formula for the parallax from the v component:

$$\pi_v = \frac{4{\cdot}74}{V_0} \frac{\overline{v \sin D}}{\overline{\sin^2 D}}$$

was employed, as, for reasons previously assigned,* the use of the τ component is not suitable. It seemed preferable to reduce the v components to those corresponding to an adopted magnitude m_0, approximately the mean magnitude of the group. The solar velocity used was $V_0 = 19{\cdot}09$ km./sec., the velocity obtained in the general or combined solution. Table VIII gives the results of the determinations and is divided into three sections containing three separate determinations of the mean parallax, the first one from the whole 881 stars, while the other two are from 839 of these stars whose proper motions have probable errors less than ± 0″·015.

The first column of the table gives the limits of the 13 groups in type and magnitude, and the second the adopted magnitude m_0, of the group to which all three parallaxes correspond. The third column contains the numbers of stars in each group, and the fifth the numbers after those with probable errors greater than ± 0″·0149 had been excluded. The fourth column gives the mean parallax from all the stars, and the sixth after exclusion of those of low weight. In neither of these mean parallaxes, however, has the effect of galactic rotation been taken into account, and this is necessary if the best results are to be obtained, as has been pointed out by Oort and Lindblad. This disturbing effect may be eliminated by dividing the sky up into a number of equal longtitude intervals and combining the parallactic motion of opposite areas with equal weights, irrespective of the number of stars. The intervals chosen were 30° and, for example, the average parallactic motions of region 0° to 30° and of region 180° to 210° were combined

* *Pub. D.A.O.*, 5, 203, 1933.

TABLE VIII

Mean Parallaxes and Absolute Magnitudes

Type and Mag. Limits	m_0	All p.m.		p.e. $\lessgtr$ 0″.0150		Correction for Galactic Rotation		
		No.	$\bar{\pi}$	No.	$\bar{\pi}$	$\bar{\pi}$	$\overline{M}$	$\overline{M}$
			″		″	″		
O	5·5	68	0·00176	55	0·00124	0·00123	−4·05	−4·05
B0–B1 < 6·0	4·0	56	370	55	369	318	−3·49	
> 5·99	6·5	38	80	33	96	106	−3·38	−3·45
B2 < 6·0	4·5	52	350	52	350	372	−2·63	
> 5·99	6·5	52	139	50	127	122	−3·07	−2·85
B3–B4 < 4·5	3·5	57	953	57	953	1013	−1·47	
4·5–5·49	5·0	122	584	119	583	512	−1·45	
5·5–6·49	6·0	107	285	105	286	332	−1·39	
> 6·49	7·0	62	206	58	201	170	−1·85	−1·50
B5–B7 < 4·5	4·0	39	953	39	953	848	−1·36	
4·5–5·49	5·0	87	680	85	686	652	−0·93	
5·5–6·49	6·0	77	441	74	438	459	−0·69	
> 6·49	7·0	64	0·00200	57	0·00198	0·00216	−1·33	−1·01

with equal weights, and the simple mean adopted with a weight equal to the total number of stars in the two opposite regions.

The differences between the parallaxes by the two methods are not large, but, when reduced to absolute magnitudes, the scatter in the different magnitude groups in the type-groups is considerably less when galactic rotation is taken into account, and there is no doubt these values are preferable. The mean absolute magnitudes of the four type-groups of the last column may be compared with other determinations in Table IX. Part of the table has been repeated from previous work,* but a few words of explanation are needed for the other values. The second column is the mean of Charlier's and Robb's values, while Oort's † values are obtained by computing the absolute magnitudes corresponding to his mean parallaxes. Using Strömberg's ‡ recent work on B-type stars brighter than 6·0 mag., $\overline{M}$ is the average absolute magnitude for all the stars, and M the maximum of the luminosity curve of the main sequence stars. The next to the last column contains the absolute magnitude obtained from the intensity of the interstellar lines,§ and the last the values just obtained from the mean parallaxes. The higher absolute magnitudes derived from the interstellar lines are undoubtedly due to the increasing strength of stellar calcium in the cooler stars blending with the interstellar lines. It should be remembered that the final Victoria absolute magnitudes in the last column are based on

* *M.N.*, **90**, 621, 1930.
† *B.A.N.*, **4**, 82, 1927.
‡ *Mt. W. Cont.*, **19**, 340, 1932.
§ *Pub. D.A.O.*, **5**, 200, 1933.

TABLE IX

Principal Values of Absolute Magnitudes

Type	Charlier, Robb	Adams	Edwards	Gerasimovič		Oort	Strömberg		P and Pe	
				$m < 4{\cdot}5$	$m > 4{\cdot}5$		$\bar{M}$	M	Ca^+	Stars
O	...	...	...			−2·9	...		−4·1	−4·0
Bo	−3·2	−3·1	−3·4						−3·8	
B1	−2·8	−2·5	−2·6	−4·0	−2·1	−2·8	−3·3	−2·9	−3·5	−3·4
B2	−2·1	−1·8	−1·9						−3·1	−2·8
B3, B4	−1·2	−1·2	−1·4	−2·1	−0·8	−1·3	−1·9	−1·4	−2·5	−1·5
B5–B7	−0·9	−0·6	−0·7	−1·3	−0·5	...	−1·6	−1·4	...	−1·0

the proper motions of over 300 more stars than the earlier determinations, and our confidence in these values should be strengthened by the regular decrease with decreasing temperature.

The Value of A.—The rotational constant A, the change in velocity produced by the galactic rotation in stars at one parsec distance from the Sun, can obviously be obtained by dividing the rotational term $\bar{r}A$ for any group of stars by the mean distance of the group. The four values of $\bar{r}A$ for the four distance-groups may be used, or new values of $\bar{r}A$ may be simply computed for a larger number of groups with a smaller range in distance by assuming the K term, the solar motion and the mean direction to the centre obtained from the general solution and solving for $\bar{r}A$ only.

The determination of the mean distance $\bar{r}$ of these various groups is more troublesome, for, though the mean parallax of any distance-group may be readily obtained by combining the mean parallaxes of the type and magnitude components, or preferably by using the mean absolute magnitudes of the several types given in the last column of Table VIII, there still remains the difficulty of determining the mean distance, $\bar{r}$, from the mean parallax, $\bar{\pi}$. As is well known, when the distance-groups are arranged on the basis of apparent magnitude, the mean distance $\bar{r}$ will be greater than the reciprocal of the mean parallax by a factor depending on the dispersion in absolute magnitude. In his work Oort * assumed a dispersion of $1^{m}{\cdot}0$ giving $\bar{r}\bar{\pi} = 1{\cdot}24$ from the formula

$$\bar{r}\bar{\pi} = e^{(0{\cdot}461\sigma)^2},$$

while Lindblad,† assuming a dispersion slightly greater than $1^{m}{\cdot}0$, obtained $\bar{r}\bar{\pi} = 1{\cdot}27$.

It seemed preferable in this case to see if the dispersion in absolute magnitude could not be derived directly from previous work on the luminosity function of the O- and B-type stars. In the work on the motions of the interstellar matter, the individual distances and hence the absolute magnitudes of 286 O- and B-type stars were derived from the intensities of the interstellar K lines in these stars and the dispersion in each spectral

* *B.A.N.*, **4**, 81, 1927. † *M.N.*, **90**, 508, 1930.

division calculated, and this is shown in the first section of Table X. This dispersion includes, however, the accidental error in the estimation of intensities, as well as other errors due to local irregularities in the distribution of the interstellar matter, the blending of stellar calcium with the interstellar lines, etc. An estimate of the total dispersion in the intensity of the K line was previously obtained.* When this dispersion is reduced to dispersion in magnitude, the weighted mean value becomes $\pm 0^{m}.64$. The weighted mean of the various type values of Table X is $\pm 1^{m}.31$, and hence the true dispersion is $\pm 1^{m}.15$.

TABLE X

Dispersions in Absolute Magnitude

Interstellar Motions			Kapteyn			Strömberg		
Type	No.	σ	Type	No.	σ	Type	No.	σ
O	59	1·33	B1	15	1·34	Bo–B2	124	1·28
Bo	45	1·28	B2	35	1·41	B3	246	1·23
B1	26	1·35	B3	120	1·17	B5	152	1·23
B2	59	1·24	B5	44	1·34			
B3–B7	97	1·34						

A check on this dispersion can be obtained from Kapteyn's † value of the absolute magnitudes of the southern helium stars. Computations of the dispersion in absolute magnitudes from his lists for four spectral subdivisions are given in the second section of the table, and a similar computation from Strömberg's table ‡ yielded the dispersions in the last section. Kapteyn's estimate of the uncertainty present in his determination is his value of $r_{\pi/\pi}$ which is about 0·25 on the average, corresponding to a dispersion of about $\pm 0^{m}.50$. The residual dispersion for Kapteyn is then $\pm 1^{m}.15$, and assuming the same error in Strömberg's value, this becomes $\pm 1^{m}.14$. The agreement between the three values of the dispersion looks almost too good, but we can hardly be far wrong in accepting it as $\pm 1^{m}.15$ which makes $\bar{r}\bar{\pi} = 1.33$, somewhat higher than Oort's and Lindblad's value.

Values of A.—The values of $\bar{r}A$ from the general solution given in the second column of Table III will serve as one set for determining A as soon as the mean parallaxes of the four groups are derived. They cover, however, a considerable range in magnitude and spectral class, and it seemed desirable to obtain as well values of $\bar{r}A$ for smaller and more homogeneous groups. It would be desirable to obtain values of $\bar{r}A$ corresponding to the 13 groups of Table VIII, for which the mean parallaxes have been obtained, but the numbers of stars are in many cases too small to give reliable values of $\bar{r}A$.

* *Pub. D.A.O.*, **5**, 209, 1933.
† *Mt. W. Cont.*, **4**, 339, 1914.
‡ *Ibid.*, **19**, 339, 1932.

Consequently, the 849 stars with usable radial velocities were arranged into 7 groups as shown in Table XI, and the values of $\bar{r}A$ determined from each of these groups from the simplified equation :

$$\bar{r}A = \sin 2(l - 324^\circ{\cdot}4),$$

the solar motion, K term, and mean direction to the centre of the combined solution, being assumed. These values with their probable errors, the numbers of stars and their mean magnitudes being given in the seventh, third and fourth columns of Table XI.

The mean parallaxes of these groups are given in the fifth column of the table and were obtained, in a somewhat involved way, from the mean absolute magnitudes of the spectral divisions in the last column of Table VIII. It was necessary to get the mean apparent magnitudes of the stars having usable radial velocities in the 13 groups of Table VIII, rather than the mean magnitudes of the proper motions as previously, and to derive revised mean parallaxes corresponding to these mean apparent magnitudes from the mean absolute magnitudes. These were then combined in an obvious way to form the mean parallaxes of the 7 groups in the fifth column of the upper part of Table XI, and the 4 groups of the combined solution in the lower part.

TABLE XI

Values of A

Group	Type and Magnitude	No.	$\bar{m}$	$\bar{\pi}$	$\bar{r}$	$\bar{r}A$	A
				"			
1	O	62	6·62	0·00089	1493	+19·95 ± 2·17	+·0134
2	Bo–B2 < 6·0	92	4·29	·00329	404	+ 9·06 ± 1·53	·0224
3	Bo–B2 > 5·99	106	6·79	·00105	1267	+15·82 ± 1·37	·0125
4	B3–B4 < 5·5	157	4·60	·00629	211	+ 2·47 ± 0·93	·0117
5	B3–B4 > 5·49	179	6·54	·00258	515	+ 4·34 ± 1·23	·0084
6	B5–B7 < 5·5	115	4·67	·00749	177	+ 2·35 ± 1·47	·0133
7	B5–B7 > 5·49	138	6·60	·00308	432	+ 2·47 ± 1·35	·0057
1	O < 5·0 ; Bo–B2 < 6·0	101	4·20	·00333	400	+ 9·74 ± 1·59	·0244
2	O > 4·99 ; Bo–B2 > 5·99	159	6·74	·00094	1410	+18·00 ± 1·24	·0128
3	B3–B7 < 5·5	272	4·63	·00681	195	+ 3·44 ± 1·33	·0176
4	B3–B7 > 5·49	317	6·56	·00280	475	+ 3·33 ± 0·86	·0070

The values of the mean distance in parsecs, $\bar{r}$ in the sixth column, are obtained by dividing 1·33 by the mean parallax, and A is then obtained from the $\bar{r}A$ of the seventh column. There is considerable dispersion in the values of A, the principal offenders, as pointed out earlier, being the brighter O to

B2 stars and the fainter B3 to B7 stars. As the first gives too large and the second too small a value of A, they tend to balance out in taking the mean.

The difficulty seems mainly to be in the determination of $\bar{r}A$, and hence the weights to be assigned in determining the mean value of A should depend on the weights of $\bar{r}A$ rather than on the number of stars. The greater the magnitude of $\bar{r}A$ the less will the irregular random motions affect its determination. Consequently, the individual values of A were weighted according to the magnitude of $\bar{r}A$. The weighted mean values of A for the two sections of Table XI are:

From the upper section—7 groups, $A = +\cdot0138 \pm \cdot0011$.
From the lower section—4 groups, $A = +\cdot0162 \pm \cdot0022$.
From stars with interstellar matter,* $A = +\cdot0166 \pm \cdot0017$.

The difference in the value of A obtained from the two methods of grouping the 849 stars is somewhat difficult to account for, but it is probably due to the method of division of the stars into groups, which gives a larger proportionate number of low values in the first section, where there are 4 out of 7 as compared with 2 out of 4 in the last. It seemed best, however, in getting the mean value of A to give the three values equal weights of the determinations, two being from 849 stellar and one from 300 interstellar radial velocities. The mean value of A therefore becomes

$$A = +\cdot0155 \pm \cdot0009 \text{ km./sec. parsec.}$$

The principal determinations of the rotational constant A are collected for comparison in Table XII below.

Precessional Corrections.—The proper motions of the P.G.C. are based upon the equinox and precessional constants adopted by Newcomb in 1874 and called by him N_1. They were derived from the right ascensions of the Equatorial Fundamental Stars in Washington Observations for 1870, and rest upon the assumption that the stellar sphere as a whole does not rotate about the Sun, and that the motions of the stars, except for the parallactic drift due to the Sun's motion, are at random. Newcomb obtained corrections to his equinox N_1 from the planets, and the Sun, and showed that a positive correction to N_1 was necessary. Similar corrections have been found from the proper motions by Kapteyn, Dyson and Thackeray, Lewis Boss, Fotheringham, and others. A new method of obtaining these corrections was developed by Oort † in 1927 in his investigations of the rotation of the galactic system. This method was based on the assumption that the systematic motions perpendicular to the galaxy will be much smaller than those in the galactic plane, and the precessional corrections were hence determined from μ_b, the proper motion components in galactic latitude. It is Oort's method which has been used in this discussion.

Our analysis was limited to the 839 proper motions with probable errors less than $0''\cdot015$, and was further restricted to stars within $\pm 20°$ of the galactic plane. As 122 stars are in higher latitudes, there remained 717 proper motions for analysis. As these very remote O- and B-type stars have

* *D.A.O.*, 5, 212, 1933. † *B.A.N.*, 4, 84, 1927.

very small and relatively uncertain proper motions, all known systematic motions must be eliminated before the minute precessional corrections can be derived. In the determination of the mean parallaxes, Raymond's correction to the proper motions in declination and the precessional corrections suggested by Boss were applied, and there are therefore available the corrected μ_α and μ_δ of every star. The other known systematic effects, remaining in the proper motions, are the components of the stream motion in the southern sky, the effects of the solar motion and the unknown correction to the precession which is being sought. The first was removed from the μ_α and μ_δ of the 139 individual members of the stream, but correction for the solar motion was postponed until the stars were arranged into type and longitude groups.

The 717 O to B7 stars were arranged into three type-groups of 225 O to B2, 295 B3 to B4, and 197 B5 to B7 stars, and each of these type-groups was divided into 18 longitude groups, each group extending over 20° in longitude and 40° in latitude. The various steps necessary in correcting for the solar motion and deriving the mean μ_b for each of the 54 groups are as follows :—

(*a*) Form the means $\bar{\alpha}$, $\bar{\delta}$, $\bar{l}$, $\bar{b}$, $\bar{\mu}_\alpha$, $\bar{\mu}_\delta$, $\bar{\mu}$, $\bar{\psi}$, $\bar{D}$, $\bar{\eta}$ and $\bar{m}$ for each of the 54 subgroups.

(*b*) Compute the $\bar{\tau}$ and $\bar{v}$ components of the proper motion.

(*c*) Obtain the mean parallax of each of the 54 groups from the apparent magnitudes and the mean absolute magnitudes of the corresponding types.

(*d*) Transform the solar motion of the general solution into proper motion by the corresponding mean parallax, and subtract it from the $\bar{v}$ component, obtaining the corrected $\bar{v}'$ for the 54 groups.

(*e*) From $\bar{\tau}$ and $\bar{v}'$ obtain $\bar{\mu}'$ and $\bar{\psi}'$, and hence

(*f*) Obtain the corrected $\bar{\mu}_\alpha'$ and $\bar{\mu}_\delta'$.

(*g*) Transform the corrected or peculiar equatorial proper motions $\bar{\mu}_\alpha'$, $\bar{\mu}_\delta'$ into galactic $\bar{\mu}_l'$, $\bar{\mu}_b'$ by means of the parallactic angles $\bar{\eta}$.

The equation of condition connecting $\bar{\mu}_b$, the corrected proper motions in galactic latitude, with the precessional corrections is

$$\bar{\mu}_b = \Delta p \sin \epsilon_1 \cos (\bar{l} - l_1) + (-\Delta e - \Delta\lambda) \sin \epsilon_2 \cos (\bar{l} - l_2), \qquad (9)$$

where

Δp = correction to the annual precession on the ecliptic,
Δe = correction to the motion of the equinox,
$\Delta\lambda$ = correction to the planetary precession,
$\epsilon_1 = 60°{\cdot}6$, the inclination of the ecliptic on the plane of the galaxy,
$\epsilon_2 = 62°{\cdot}0$, the inclination of the equator on the plane of the galaxy,
$l_1 = 153°{\cdot}3$, gal. long. of the ascending node of the ecliptic on the galaxy,
$l_2 = 180°$, gal. long. of the ascending node of the equator on the galaxy.

While the stars were divided into three type-groups, this was mainly on account of the large spread in absolute magnitude, and hence in distance, between the O to B2 stars and the later types, which would tend to make the mean parallaxes and the solar motion corrections less trustworthy. In solving the precessional corrections, the weighted means of the three values

for each of the 18 longitude groups were used, the scatter and the curve resulting from the solution being shown in fig. 3. The 18 observation equations and the normal equations were formed in the usual way and resulted in the values :

$$\Delta p = +0''\cdot0038 \pm 0''\cdot0012,$$
$$(-\Delta e - \Delta\lambda) = -0''\cdot0011 \pm 0''\cdot0012,$$

which it should be remembered are corrections to Boss's suggested precessional corrections and not to Newcomb's precession.

Oort * has suggested, owing to changes in the value of the mass of Venus, that Newcomb's planetary precession $\Delta\lambda$ should have a correction of $-0''\cdot0020$, and when this is applied to the above values and they are reduced to the usual form for direct application to μ_α and μ_δ they become

$$\Delta\mu_\alpha = +0^s\cdot00016 - 0^s\cdot00010 \sin\alpha \tan\delta,$$
$$\Delta\mu_\delta = -0''\cdot0015 \cos\alpha,$$

the corrections to be applied to the proper motions already corrected by Boss's suggested precession. For comparison with Boss's and Oort's values of the precessional corrections to Newcomb's precession the above values were transformed to corrections to Newcomb and all three are given below.

Corrections to Newcomb's Precession

Boss, $\Delta\mu_\alpha = +0^s\cdot00021 - 0^s\cdot00015 \sin\alpha \tan\delta$,
$\Delta\mu_\delta = -0^s\cdot0023 \cos\alpha$;

Oort, $\Delta\mu_\alpha = +0^s\cdot00009 - 0^s\cdot00030 \sin\alpha \tan\delta$,
$\Delta\mu_\delta = -0''\cdot0045 \cos\alpha$;

P and Pe, $\Delta\mu_\alpha = +0^s\cdot00005 - 0^s\cdot00025 \sin\alpha \tan\delta$,
$\Delta\mu_\delta = -0''\cdot0038 \cos\alpha$.

The three determinations are from different data and there are some differences in method. Boss employed 5413 stars uniformly distributed over the sky, and from the observed proper motions derived values for the solar motion and the precessional corrections. His solar apex was $\alpha = 270°$, $\delta = +34°$, some 6° higher than derived from radial velocities. This was due, as we now know, to errors in μ_δ which may be removed by Raymond's correction. Oort applied this correction to the proper motions of the 5413 stars, removed the solar motion, and derived his precessional corrections from the galactic latitude components of the corrected or peculiar proper motions. He also obtained precessional corrections from part of this material, from 771 distant stars, 441 B-type and 330 Cepheids, "*c*," "*n*" and planetaries, highly concentrated to the galactic plane, in practical agreement with those from the 5413 stars. Our solution differs from Oort's in that the material is more homogeneous, 717 O- and B-type stars as compared with 441 O and B stars and 330 other types, and that a correction for stream motion among the southern stars was applied. Fig. 3, upper section, gives the curve representing the precessional correction obtained from the solution of the

* *B.A.N.*, **4**, 85, 1927.

corrected latitude components of the proper motions, with the observations plotted.

The Rotational Constant B.—As previously indicated the constant B can

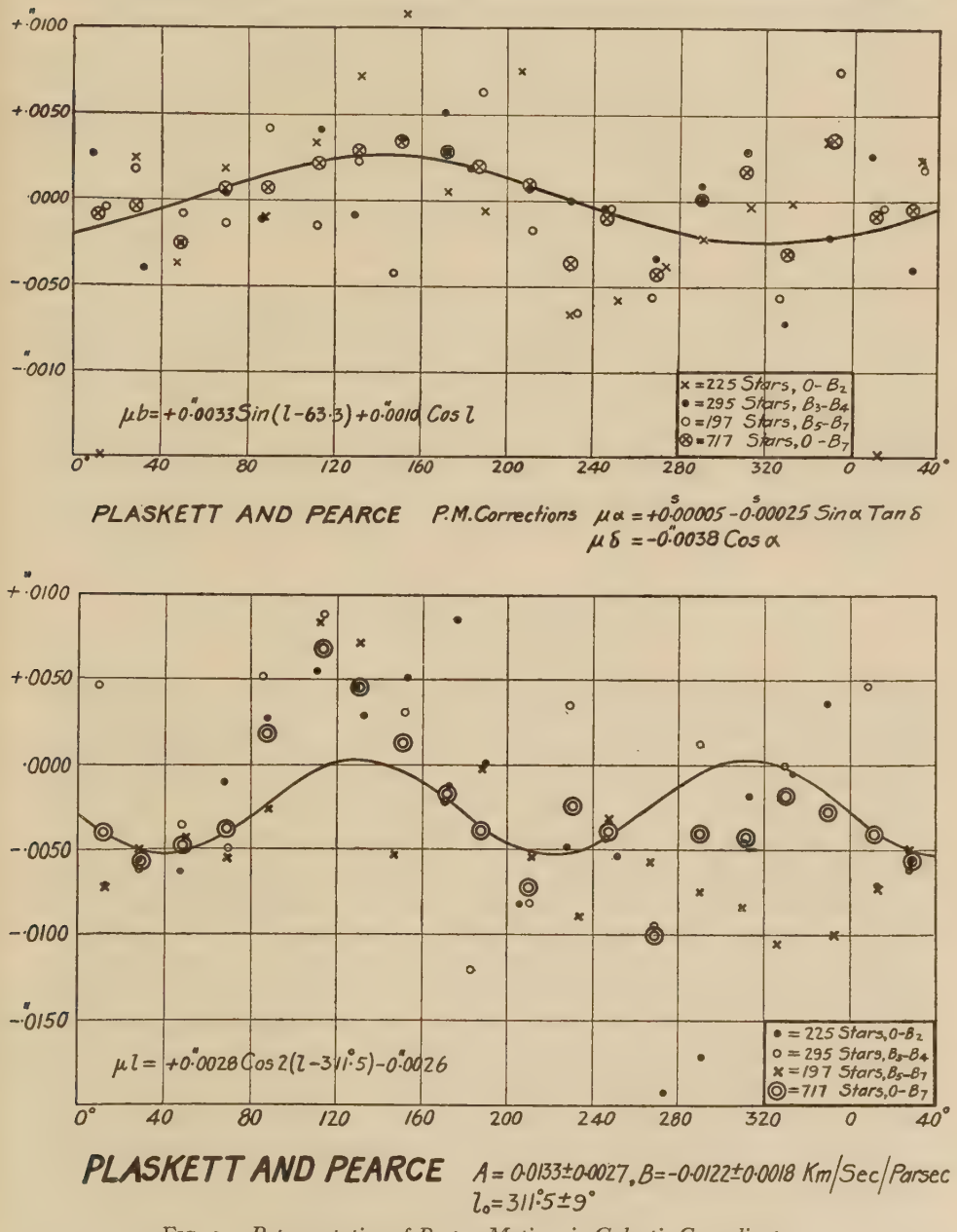


FIG. 3.—*Representation of Proper Motions in Galactic Co-ordinates.*

be obtained from the components in galactic longitude of the proper motions. Consequently, to the corrected $\bar{\mu}_\alpha'$ and $\bar{\mu}_\delta'$ used in obtaining the precessional corrections, our corrections to Boss's precessional corrections were applied,

and the resulting equatorial were transformed into galactic proper motions. The longitude components thus obtained are plotted in the lower section of fig. 3, with the rotational curve derived from the solution.

The equation of condition used for obtaining B is

$$\bar{\mu}_l' = \frac{A}{474} \cos 2(\bar{l} - l_0) \cos \bar{b} + \frac{B}{474} \cos \bar{b}. \qquad (10)$$

The least-squares solution of this equation with three unknowns A, B and l_0 gives the following values :—

$$A = +0{\cdot}0133 \pm {\cdot}0027 \text{ km./sec.}$$
$$B = -0{\cdot}0122 \pm {\cdot}0018 \text{ km./sec.}$$
$$l_0 = 311^\circ{\cdot}5 \pm 9^\circ{\cdot}1.$$

These values of A and l_0 agree as well as could be expected, considering the relative uncertainty of the proper motions, with the corresponding values from the radial velocities of $A = +0{\cdot}0155 \pm 0{\cdot}0009$ km./sec. and $l_0 = 324{\cdot}4 \pm 2{\cdot}4$, which are of much greater weight. There can be no doubt that the radial velocity values are preferable, and consequently the latter values of A and l_0 were substituted for the corresponding unknowns in the equation, and a new value of B of greater weight obtained from a new solution with only B as the unknown quantity. We have, therefore,

$$A = +0{\cdot}0155 \pm {\cdot}0009 \text{ km./sec.}$$
$$B = -0{\cdot}0120 \pm {\cdot}0027 \text{ km./sec.}$$
$$l_0 = 324^\circ{\cdot}4 \pm 2^\circ{\cdot}4,$$

as the final values from the radial velocities of 849 and the proper motions of 717 O to B7 stars. While Oort's value from the 771 distant stars concentrated to the galactic plane is just twice this value, his determination from the 5413 Boss stars is $B = -0{\cdot}0109 \pm {\cdot}0023$ km./sec., practically the same as the above.

For comparison, the most representative values of the constants A and B are collected together in Table XII. The values of A, tabulated first, show excellent agreement, but the values of B have a considerably greater range, with six smaller and three larger values, the simple means being respectively $-0''{\cdot}0049$ and $-0''{\cdot}0024$. It is interesting to note that, generally speaking, except in those determined here, the smaller values come from analysis of large numbers of stars of all spectral classes, while the larger values are from limited groups of stars. The reason why the values from the limited groups are larger may be tentatively assigned to the presence of the southern stream among the B-type stars. When this is eliminated, as in the present analysis, the value of B thus obtained is similar to those from the groups of several thousand stars, where the effect of 139 stream motions is so diluted as to be inappreciable, although giving large values in the limited groups of about one-sixth the numbers of stars. The value of B of $-0''{\cdot}0120$ km./sec./parsec, the last value in the table, obtained from the analysis of 717 proper motions and 849 radial velocities of the O to B7 stars, is therefore adopted. This procedure is consistent with that followed

Table XII

Values of Rotational Constants

Author and Material	Analysis of	No.	A (km./sec.)	$B/4{\cdot}74$
Oort, O–B5, "c," A–G, etc.	Rad. Vel.	700	$+{\cdot}019 \pm {\cdot}002$	
Plaskett, O–B5	,,	450	$+{\cdot}0155 \pm {\cdot}0007$	
Pearce, O–B5	,,	680	$+{\cdot}016 \pm {\cdot}001$	
P and Pe, Interstellar Calcium	,,	314	$+{\cdot}0166 \pm {\cdot}0017$	
P and Pe, O–B7	,,	849	$+{\cdot}0155 \pm {\cdot}0009$	,,
Charlier, Galactic Boss Stars	Prop. Mot.	4182	...	$-0{\cdot}0024$
Fotheringham	,,	...	...	$-0{\cdot}0015$
Oort	,,	5413	...	$-0{\cdot}0023 \pm {\cdot}0007$
Oort	Both	771	$+{\cdot}019 \pm {\cdot}002$	$-0{\cdot}0050 \pm {\cdot}0011$
Raymond and Wilson, Space Motions	,,	4233	$+{\cdot}015$	$-0{\cdot}0034$
Dyson, Bo–Ao Boss Stars	Prop. Mot.	1281	$+{\cdot}012$	$-0{\cdot}0046$
Lindblad, F–K	Both	714	$+{\cdot}015 \pm {\cdot}005$	$-0{\cdot}0051$
P and Pe, O–B7	Prop. Mot.	717	$+{\cdot}0133 \pm {\cdot}0023$	$-0{\cdot}0026 \pm {\cdot}0006$
P and Pe, O–B7	Both	849	$+{\cdot}0155 \pm {\cdot}0009$	$-0{\cdot}0025 \pm {\cdot}0006$

throughout, where as soon as the value of any constant was finally obtained, it was used in the subsequent work without reference to other determinations.

Dimensions of the Galactic System and Comparison with Other Systems

Dynamical Constants of the Galaxy.—In order to obtain the distance to the gravitational centre of the galaxy, there is needed, in addition to the rotational constants A and B, just derived and given in the last line of Table XII, the rotational velocity, V, in the neighbourhood of the Sun. This velocity can only be obtained from analysis of the radial velocities of external objects which do not share in the galactic rotation. The system of globular clusters, which is little flattened to the galactic plane and not at all when space absorption is considered, probably does not participate in the rotation, and, therefore, may be used to determine the velocity V. Similarly, the external galaxies almost certainly have no rotational motion, and an analysis of their radial velocities, though complicated by their recessional velocities, should also give V.

The solar motion as determined from the globular clusters by Strömberg * is 286 km./sec. towards $\alpha = 306°$, $\delta = +62°$, and this, corrected for the motion of the Sun with respect to the nearby stars, gives a rotational velocity of the stars of 272 km./sec. in the Sun's neighbourhood. If there is rotational motion of the clusters, then the true rotational velocity would be increased. Similarly, the solar velocity from the radial velocities of 24 extra-galactic nebulæ, assuming that the nebular velocities of recession vary directly with the distance, was calculated by Hubble † as 280 km./sec. Oort ‡ accepts

* *Mt. W. Cont.*, **13**, 214, 1925. † *Proc. Nat. Ac. Sc.*, **15**, 168, 1929.
‡ *B.A.N.*, **3**, 281, 1927.

V as 272 km./sec., while Lindblad * adopts 275 km./sec. We shall adopt here the rounded value of $V = 275$ km./sec.

The simple formula connecting R, V, A and B is

$$V/R = A - B, \tag{11}$$

which, when $A = +\cdot0155$ km./sec./parsec, $B = -\cdot0120$ km./sec./parsec, $V = 275$ km./sec., gives $R = 10{,}000$ parsecs and the period of rotation at the Sun of 224,000,000 years. It has usually been assumed, from the distribution of the globular clusters, that the Sun is situated at about two-thirds of the distance from centre to edge, which gives the total diameter of the galaxy as 30,000 parsecs. Lindblad † by a theoretical treatment of his rotational equations, in combination with data from Strömberg's ‡ discussion of asymmetry, computes the sun to be 0·77 of the distance from the centre to the effective limit of the system, and this would make the diameter of the galaxy 26,000 parsecs. Any reasonable representation of the limits of the system from the distribution of the globular clusters, drawn of elliptical form in Shapley's § diagram, and circular in Stebbins',‖ points to the ratio of distance of the Sun from the centre to the edge of the system as much nearer 0·67 than 0·77. Moreover, Trumpler's plot ¶ of the distribution of the open clusters, when the distances are corrected for space absorption, shows that in the direction away from the centre, longitude 145°, the open clusters extend to about 5000 parsecs from the Sun. There seem, therefore, good reasons for assuming the Sun to be about two-thirds of the distance from centre to edge, and the diameter of the galaxy as 30,000 parsecs.

The distribution of mass in the galaxy may reasonably be assumed such as to produce a total gravitational force K, formed of the sum of two forces K_1 varying inversely proportional to R^2, and K_2 varying proportionally to R. Oort's ** formula gives the ratio

$$K_1/K = 4A/3(A - B) = 0{\cdot}75, \tag{12}$$

indicating that three-fourths of the total gravitational force is due to concentration of the matter towards the centre of the galaxy. Lindblad †† gives formulæ to determine the total mass of the system where M_1 is the mass concentrated at the centre and M_2 the mass of a uniformly distributed flattened spheroid of stars and diffuse matter of the diameter of the galaxy.

$$\begin{aligned} M_1 &= 4\pi R^3/3G\omega A, \\ M_2 &= 4\pi R_1{}^3/3\pi G\omega(\omega - 4/3A), \end{aligned} \tag{13}$$

R being the distance of the Sun from the centre, and R_1 the radius of the system. These formulæ give, when the constants adopted here are used, the following values :—

$$M_1 = 263 \times 10^{42} \text{ gm.} \qquad M_2 = 67 \times 10^{42} \text{ gm.}$$
$$M_1 + M_2 = 330 \times 10^{42} \text{ gm.} \qquad = 16{\cdot}5 \times 10^{10} \odot.$$

* *Handbuch der Astrophysik*, 5, 1067.
† *Loc. cit.*, p. 1051.
‡ *Mt. W. Cont.*, 13, 205, 1925.
§ *Star Clusters*, p. 174.
‖ *Proc. Nat. Ac. Sc.*, 19, 226, 1933.
¶ *L.O.B.*, 14, 179, 1930.
** *B.A.N.*, 4, 88, 1929.
†† *Loc. cit.*, p. 1068.

We, therefore, have for the dimensions of the galaxy dynamically determined from analyses of the motions of the O- and B-type stars :

Distance from Sun to centre	R	$=10{,}000$ parsecs.
Diameter of galaxy	D	$=30{,}000$ parsecs.
Rotational velocity at Sun	V	$=275$ km./sec.
Period of rotation at Sun	P	$=224{,}000{,}000$ years.
Ratio of gravitational forces	K_1/K	$=0{\cdot}75$.
Total mass of galaxy	M	$=16{\cdot}5\times10^{10}\odot$.

The value of the mass derived is considerably larger than usually assigned to the galactic system, which has been estimated by various authorities to contain 3×10^{10}, 10×10^{10}, or even 20×10^{10} stars. Owing to the presence of absorbing matter these estimates are uncertain, but if we assume there are 10^{10} stars with the average mass of the Sun in the system, these will provide little more than half the mass required for the distance and rotational velocity derived. This, however, takes no account of the absorbing matter between the stars which will not only act to reduce the estimates of the numbers of stars in the galaxy, but will also add very materially to the mass of the system. Pannekoek * computes that a sphere 6000 parsecs in diameter with Eddington's assumed density of interstellar matter of 6×10^{-23} gm./cm.^{3} would have a mass 10×10^{10} that of the Sun more than sufficient to make up the mass required. If we assume that this matter is distributed in a cylindrical layer 1000 parsecs thick and the diameter of the galaxy, 30,000 parsecs, which seems more in accord with present knowledge of the distribution of the diffuse matter, and if we further assume only one-sixth the density adopted by Pannekoek, or $\rho=10^{-23}$ gm./cm.^{3}, this will give a slightly greater mass than 10×10^{10} Suns. Hence, such a relatively thin layer of absorbing matter only requires a density of 10^{-23} in order to provide more than sufficient mass to give the rotation observed.

The Diameter of the Galactic System.—Previous determinations of the distance, R, to the centre of the galaxy by rotational methods have given smaller values. In his first analysis Oort † obtained 5900 and later 6300 ‡ parsecs, while Lindblad § gave a preliminary value of 6500 parsecs, and subsequently, by combining several determinations of the constant B, obtained a distance of 9400 ‖ parsecs to the centre of the galaxy. Wilson ¶ from an analysis of the space motions of 4233 stars obtained 8800 parsecs as the distance to the centre. The tendency has been for the values of R to increase with improved material, and the value of 10,000 parsecs obtained here is not out of step with other dynamical determinations. It will be of interest to give a brief summary of the values obtained by other methods, particularly as Oort's distance of about 6000 parsecs was so much smaller than the 20,000 to 30,000 parsecs obtained from the luminosity distances of the globular clusters that there seemed little hope of reconciling this disparity.

* *B.A.N.*, **4**, 40, 1923.
† *B.A.N.*, **3**, 281, 1927.
‡ *Idem, ibid.*, **4**, 88, 1927.
§ *M.N.*, **90**, 513, 1930.
‖ *Handbuch der Astrophysik*, **5**, 1067.
¶ *A.J.*, **40**, 121, 1930.

Until some fifteen years ago, the only methods available depended on the analyses of the star counts, proper motions, parallaxes, luminosities, etc., on the assumption that the galaxy was a flattened cluster of stars surrounding the Sun and gradually thinning out towards the edges. The diameters given varied from 2000 to 5000 parsecs, although Kapteyn later derived a value of the order of 15,000 parsecs. These methods were, however, superseded by Shapley's work on the globular clusters about 1918. He estimated the distances of these objects by luminosity methods based primarily on the period-luminosity relation of the Cepheids and plotted their distribution with respect to the galactic plane. He reached the conclusion from their symmetrical distribution, that they were members of the galaxy which was concentric and coterminous with them, and this conclusion has been generally accepted. This made the diameter of the galactic system about 90,000 parsecs with a distance from the Sun to the centre of 30,000 parsecs, three times the gravitational distance just derived.

The estimated diameter reached a maximum in this determination and has been decreasing ever since, as later work by Wilson, Gerasimovič and others changed the zero point of the period-luminosity relation and reduced the luminosity distances by about 35 per cent. This made the distance and diameter 20,000 and 60,000 parsecs respectively, which, however, are still twice the gravitational dimensions. Shapley * has recently revised his cluster distances, and hence the galactic dimensions, the distance from the Sun to the centre being now 16,000 parsecs, 60 per cent. larger than the dynamical distance. This disparity, while not so bad as the early ratio of five to one, is still too large to be accepted, and some disturbing factor in one of the methods must be sought.

This now appears to be due to the presence of absorbing material near the central plane of the galaxy, which will obviously cause distant stars to appear fainter, and hence their distance estimated as too great. While various references to the possibility and effects of absorption had been made, it was generally believed to be practically negligible, and it was not until Trumpler's† work in 1930 on the open clusters and his determination of the absorption near the galactic plane as $0^{m}.67$ per 1000 parsecs, that attention was forcibly directed to its presence and to its effect on the luminosity determinations of stellar distances. Van de Kamp ‡ has obtained estimates by ingenious methods of the effect of absorbing matter on the distance to the centre of the galaxy. His values of the luminosity distance, corrected for the effects of space absorption, range from 5500 to 13,700 parsecs, with preference for the later and smaller values. They are all, however, affected by some uncertainty in the data and assumptions employed, and a more definite determination is desirable.

This has been provided recently by Stebbins § from measures of the colour excess of the globular clusters with a photo-electric photometer

* *Star Clusters*, p. 177.
† *L.O.B.*, **14**, 154, 1930.
‡ *A.J.*, **40**, 145, 1930 ; **41**, 81, 1931 ; **42**, 97, 1932.
§ *Proc. Nat. Ac. Sc.*, **19**, 222, 1933.

attached to the 100-inch Mt. Wilson telescope. These measures have given us direct and apparently reliable determinations of the space absorption near the galactic plane, and Stebbins has determined its effect on the luminosity distance to the galactic centre. He finds that Shapley's latest distance of 16,000 parsecs is reduced, when the effect of space absorption is eliminated, to about 10,000 parsecs, and hence brings the distances from the Sun to the galactic centre, as determined by the dynamical and luminosity methods, into agreement. This agreement by two totally different and independent methods should give us confidence in both determinations and in the substantial correctness of the determinations of the diameter of the galaxy as 30,000 parsecs.

Similarity of the Galaxy to the Spiral Nebulæ.—There still remains a difficulty in accepting the diameter of the galaxy as 30,000 parsecs, namely, the disparity in size between it and the largest known external system, the Andromeda Nebula, which is about 12,000 parsecs in diameter. While originally the disparity was about 7 to 1, this has been reduced by the steps described to about 2·5 to 1. This difference is still disturbing, particularly as the Andromeda itself is some five times larger in diameter than the average extra-galactic nebula, and it has led to doubts whether the galaxy can be a single dynamical unit like the other galaxies obviously are, and to hypotheses that the galaxy is complex, a combination of nebulæ and star clouds. Two recent investigations have, however, removed this difficulty, not by reducing the size of the galaxy still further, but by showing the Andromeda to be much larger than hitherto believed.

Hubble * in 1932 provisionally identified 140 nebulous objects in photographs of the Andromeda and its neighbourhood as globular clusters, whose distribution not only appears similar to that of the clusters near the galaxy, but this distribution also indicates that the diameter of the nebula is about 30,000 parsecs, the same as that just found for the galaxy. Stebbins † has recently applied his photo-electric photometer, attached to the 100-inch telescope, to measures of sky luminosity across and near the Andromeda Nebula. He finds that the background luminosity of the sky near the nebula begins to increase definitely in regions so far beyond the extension shown in long-exposure photographs, that the effective diameter must be double that previously established. These two independent determinations agree in deriving a diameter of the Andromeda Nebula practically the same as that obtained for the galaxy, which was also derived by two independent methods, and the disparity in size has now disappeared.

The similarity, however, does not end here, as large-scale photographs of the Andromeda Nebula show stars and star clouds bright and dark diffuse nebulæ, novæ, Cepheids and globular clusters apparently identical in constitution and distribution to the same well-known objects of our galaxy. Although it is impossible to obtain from our position within the galaxy more than a vague idea of its structure, no one who examines photographs of the outer extensions of the Andromeda, and notes the similarity between the star clouds and the dark and bright nebulosity of the Andromeda and those shown

* *Ap. J.*, **76**, 44, 1932. † *Proc. Nat. Ac. Sc.*, **20**, 93, 1934.

in photographs of the Milky Way clouds, can doubt that the galaxy is a great spiral similar in general structure and dimensions to the Andromeda Nebula. That they are both dynamical units, rotating in their own planes, provides additional confirmation of their similarity. There is, hence, no necessity for considering the galaxy as a complex aggregation of small nebulæ, and our inherent belief in the homogeneity of the Cosmos is now well supported observationally.

Conclusions

Analyses of 849 radial velocities, the majority obtained at Victoria, and 881 proper motions of O to B7 stars have resulted in the following conclusions :—

1. The convergent stream velocity and radial component of the well-known stream motion among the southern B-type stars were redetermined, and both radial and transverse components of the stream motion were eliminated from the participating stars.

2. The 849 stars were arranged into four distance-groups, and a general solution made for a common solar motion and K term for all the stars and of $\bar{r}A$ and l_0 for each group.

3. The solar motion obtained was $V_0 = 19{\cdot}09 \pm 0{\cdot}57$ km./sec. towards $L_0 = 21^\circ{\cdot}4 \pm 1^\circ{\cdot}8$ and B_0 assumed as $+20^\circ$, both velocity and apex agreeing closely with the values obtained from the stars in general.

4. The K term, by correction for the stream motion of the southern stars, and by introduction of terms for the galactic rotation, has been reduced from previous value of the order of $+5$ km./sec. to $+1{\cdot}13 \pm 0$ km./sec. Calculation of the Einstein displacement for the O and B stars of average mass and density gives a mean value of $+1{\cdot}27$ km./sec., and this mysterious positive residual is now satisfactorily explained.

5. The mean value of l_0 from the four distance-groups is $324^\circ{\cdot}4 \pm 2^\circ{\cdot}4$, while the values of $\bar{r}A$ are well determined and correspond approximately to the mean distances of the groups.

6. The mean parallaxes of 13 types and magnitude groups were derived from the proper motions, the resulting mean absolute magnitudes being: Class O $= -4{\cdot}05$; Class Bo, B1 $= -3{\cdot}45$; Class B2 $= -2{\cdot}85$; Class B3, B4 $= -1{\cdot}50$; Class B5, B7 $= -1{\cdot}01$.

7. Combination of the mean parallaxes, reduced by correction for the dispersion in absolute magnitude to mean distances, with corresponding values of $\bar{r}A$, gave various values of A, the final mean value being

$$A = +0{\cdot}0155 \pm 0{\cdot}0009 \text{ km./sec./parsec.}$$

8. New precessional corrections to those suggested by Boss were derived by Oort's method from the latitude components of the proper motions of 717 stars, within galactic latitudes $\pm 20^\circ$, corrected for Raymond's correction to μ_δ, for solar motion, and stream motion. The resulting corrections to Newcomb's precession are

$$\Delta\mu_\alpha = +0^s{\cdot}00005 - 0^s{\cdot}00025 \sin\alpha \tan\delta,$$
$$\Delta\mu_\delta = -0''{\cdot}0038 \cos\alpha.$$

9. After these precessional corrections were applied to the corrected 717 proper motions they were transformed into galactic components and the longitude components were used to derive A, B and l_0, giving $l_0 = 311°{\cdot}5 \pm 9°{\cdot}1$; $A = +0{\cdot}0133 \pm {\cdot}0027$ km./sec./parsec; $B = -0{\cdot}0122 \pm {\cdot}0018$ km./sec./parsec.

10. While the values of A and l_0, considering the relative uncertainty of the proper motions, agreed satisfactorily with the radial velocity values, the latter have much greater weight, and, being assumed in the equations of condition, gave a final value of $B = -0{\cdot}0120$ km./sec./parsec.

11. The distance to the galactic centre follows from the equation $V/R = A - B$ as soon as the value of 275 km./sec. for the rotational velocity V at the Sun was adopted, giving the distance $R = 10{,}000$ parsecs and the diameter 30,000 parsecs.

12. Other dynamical consequences are: Period = 224,000,000 years. Ratio of force due to mass concentrated at centre to total mass $= 0{\cdot}75$. Total mass $= 16{\cdot}5 \times 10^{10} \odot$.

13. The most recent value of Shapley of the luminosity distance to the centre of the globular cluster system of 16,000 parsecs when corrected for space absorption by Stebbins's measures of the colour excess of the globular clusters comes out at 10,000 parsecs, in agreement with the dynamical distance.

14. The great disparity in size between the galaxy and the largest known external system, the Andromeda Nebula, of diameter 12,000 parsecs, has been practically removed by Hubble's investigation of the globular clusters around the Andromeda, giving a diameter of 30,000 parsecs, and Stebbins's photo-electric measures of the luminous extension, giving nearly the same size. Thus the difficulty of the previous abnormal size of the galaxy now disappears.

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STELLAR COLOUR TEMPERATURES AND THE ABSORPTION COEFFICIENT.

R. v. d. R. Woolley, Ph.D.

§ 1. Attention has recently been focussed on the general question whether the stars in general and the Sun in particular should or should not radiate like black bodies. The recent observations of Greaves, Davidson and Martin * have shown that the colour temperature of the Ao stars is probably as high as 18,000°. Previous estimates of the temperature of stars of this class, from ionisation data, gave a temperature of about 10,000°. While it is not certain that the ionisation temperatures are correct, the electron pressure being to some extent indeterminate, it is clear that there is some evidence suggesting that the colour temperature is far in excess of the ionisation temperature. When we turn to the case of the Sun it is possible

* *M.N.*, **94**, 488, 1934.

to measure an additional quantity, namely, the effective temperature of the body. There are, of course, many ways of specifying the temperature of a body which is not in thermodynamical equilibrium, and we have no reason *a priori* for supposing that the effective temperature (measured by the total emergent radiation per sq. cm. of the surface of the star) should be equal to the colour temperature, or to the ionisation temperature. A very full discussion of the solar radiation is given by Eddington in *The Internal Constitution of the Stars*, chap. xii. The discussion lays emphasis on Abbot's observations which indicated that the solar colour temperature was roughly equal to the effective temperature. We must in this connection pay attention to what account was taken of the Fraunhofer lines. The primary purpose of the colour investigations conducted by Abbot was the determination of atmospheric absorption coefficients to be applied to his work on the measurement of the solar constant with the pyrheliometer. For this purpose the correct procedure was undoubtedly the drawing of a curve representing as nearly as possible the mean radiation in each section of the spectrum, thus smoothing out the distinction between the continuous background and the absorption lines—which is what Abbot did. But should we not use the continuous radiation between the lines, rather than this smoothed value, to give the colour temperature? In the first instance this is merely a matter of definition, and the colour temperature may stand defined as the colour temperature of the continuous background between the lines. It remains to be argued theoretically and to be tested by experiment whether the colour temperature defined in this way is or is not equal to the effective temperature.

As a matter of direct observation the Sun's colour temperature is not equal to its effective temperature. H. H. Plaskett * measured the true colour temperature of the Sun by observing the continuous background between the absorption lines. He found for the centre of the disc $6700° < T_c < 7000°$ *K*. These observations have received support from recent work by Shajn,† who has measured the amount of light absorbed in every 25 A. of the solar spectrum and applied a correction to Abbot's curve, which brings it into agreement with Plaskett's. As is well known, Abbot's value of the effective temperature of the Sun is $T_e = 5740°$ *K*. The difference between T_c and T_e is partly due to the fact that the colour temperature of the centre of the Sun exceeds that of light integrated over the whole disc by about 400°, but even so the gap is by no means bridged and the remaining difference of about 700° cannot conceivably be attributed to observational error.

Turning to the theoretical significance of these results, it is clear from an inspection of Eddington's exposition in *The Internal Constitution of the Stars* that the relation between T_c and T_e is bound up with the variation of the coefficient of continuous absorption k_ν with the frequency ν. In particular, we may ask whether k_ν is constant (independent of ν). There was in 1926 (when the book was published) no physical reason for supposing that the absorption coefficient k_ν was independent of ν, but Milne had

* *Pub. D.A.O.*, **2**, 213, 1923. † *M.N.*, **94**, 652, 1934.

deduced that k_ν was constant from observations of darkening at the limb. Because it was not deemed likely that k_ν was constant, and because it was not then generally realised that the colour temperature was higher than the effective temperature, no emphasis was laid on a particular feature of the model with k constant, which is clearly exhibited in Table XLIII, p. 327 of *I.C.S.*, namely, that the colour temperature of the model does exceed the effective temperature by a small amount ($T_c = 6220°$ compared with $T_e = 6070°$ for the centre of the disc). When we take account of the absorption lines we find that this increase of T_c over T_e is somewhat larger, so much so as nearly to admit of the deduction that k_ν is effectively a constant over the visual spectrum.

The form of the continuous absorption coefficient k_ν as a function of the frequency ν is discussed by Pannekoek.* The continuous absorption is effected by the photo-electric ejection of electrons from the several stationary states in the parent atom. The formula for the contribution to the general absorption made by each stationary state is of form $\kappa \propto \lambda^3$ for all λ greater than a certain limit, beyond which the atoms in this state are transparent to (continuous) radiation. The summation over all the possible stationary states of all the atoms composing the solar atmosphere does not lead to a law $k \propto \lambda^3$ because the limits fall in different places. If E is the energy of the state and I the ionisation potential of the atom, then the limit falls at $h\nu = I - E$. The greater E is, the further is this limit placed towards the red end of the spectrum: and the greater the value of E the less atoms will be raised into this state by thermal agitation. Despite the λ^3 law for each state, k_ν will be larger in the violet than in the red, because the number of states which can absorb ultra-violet radiation is greater than the number which absorb red radiation, many limits falling between the two. Pannekoek derives a formula for k_ν by supposing that there is, effectively, an infinite number of atomic states. He then finds that k_ν is proportional to

$$\left(\frac{\nu}{\nu_1} - \frac{RT}{h\nu_1}\right) e^{h\nu/RT},$$

where $h\nu_1$ is the ionisation potential of the element. On this theory k is very far removed from being a constant.

This view of the state of affairs overlooks the fact that a great part of the continuous absorption in the working range of wave-length—say from 6500 A. to 4000 A.—comes from absorption in a single band, namely, the hydrogen absorption at the head of the third (Paschen) series. Over this range of wave-length the hydrogen absorption follows very nearly a λ^3 law, and the effect of a mixture of hydrogen and metal is to some extent a cancellation of the two effects of the hydrogen red absorption and the metallic absorption in the blue. The net effect may very well be the production of a k that is more or less constant throughout the working range. Since we do not know the oscillator strengths of all the bands which terminate within the working range it is impossible to calculate the relative importance of

* *M.N.*, 91, 162, 1930.

the "hydrogen" and "metallic" absorption (*any* band which has its head at a longer wave-length than 6500 A. counts as "hydrogen" absorption). We may, however, take it from Milne's work on the darkening at the limb that the mixture is such that k = constant is not a bad approximation for the Sun. We can then work out the colour temperature. The agreement of this colour temperature with the observed colour temperature supports the assumption that k is approximately constant.

Mere increase of temperature, without any increase of hydrogen content, will increase the importance of the hydrogen relative to the metallic absorption, since increase of temperature increases the number of atoms in the third hydrogen state. Stars whose effective temperatures are 10,000° should accordingly show a higher colour temperature: though it remains to be calculated whether the same proportion of hydrogen to metals as gives k constant in the Sun will give colour temperatures of 18,000° in the Ao stars. It must be mentioned that the limiting case where the atmosphere contains nothing but hydrogen has been worked out by Unsöld,* who arrives at very extraordinary conclusions. The colour temperature is not even a monotonic function of the effective temperature. Unsöld gives the result for the pure hydrogen atmosphere that when the effective temperature is 10,000° the colour temperature is 31,500° by comparison of λ 6500 with λ 5000 and 42,000° by comparison of λ 5000 with λ 4000.

The excess of colour temperature over ionisation temperature, as observed by Greaves, Davidson and Martin, becomes immediately intelligible. It is simply due to the fact that there is a good deal of hydrogen in the stars. Notice that we should expect the opposite effect, colour temperature *less* than effective temperature, in the late-type stars. If they have the same hydrogen content as early-type stars, the low temperatures keep too few atoms in the excited states to produce any appreciable absorption in the visible region, but that this effect will be to some extent offset by the greater strength of the absorption lines in the late-type stars.

§ 2. Although the analysis in chap. xii of *The Internal Constitution of the Stars* actually covers the ground in which we are interested, it is as well to repeat the analysis in such a way as to give *explicit* reference to absorption lines.

Let $\mathcal{J}_\nu(\theta)d\nu d\omega/4\pi$ be the flow per sq. cm. per sec. of radiation in frequencies from ν to $\nu+d\nu$ travelling in the solid angle $d\omega$ at an inclination θ to the vertical. Taking the axis of x vertical (positive if outwards from the surface) let ds be an element of length in the direction θ; then $ds = dx \sec\theta$. In traversing a cylinder of length ds in the direction θ the flow $\mathcal{J}_\nu(\theta)d\nu d\omega/4\pi$ will lose by absorption $(\mathcal{J}_\nu(\theta)d\nu d\omega/4\pi)\cdot k_\nu\rho ds$, where k_ν is the coefficient of absorption of radiation of frequency ν, and ρ is the density. The gain by emission of radiation in the length ds is $(j_\nu d\nu d\omega/4\pi)\rho ds$. Balancing the gains and losses we have

$$\frac{d\mathcal{J}_\nu(\theta)}{ds} = -k_\nu\rho\mathcal{J}_\nu(\theta)+j_\nu\rho,$$

* *Zs. f. Astrophys.*, **8**, 32, 1933.

or

$$\cos\theta\frac{d\mathcal{J}_\nu(\theta)}{k_\nu\rho dx}=-\mathcal{J}_\nu(\theta)+j_\nu/k_\nu. \tag{2.1}$$

Now let

$$\mathcal{J}_\nu=\frac{1}{4\pi}\int\mathcal{J}_\nu(\theta)d\omega,\qquad H_\nu=\frac{1}{4\pi}\int\mathcal{J}_\nu(\theta)\cos\theta d\omega,\qquad K_\nu=\frac{1}{4\pi}\int\mathcal{J}_\nu(\theta)\cos^2\theta d\omega.$$

Multiply (2.1) by $d\omega$ and integrate over a sphere; then multiply (2.1) by $\cos\theta d\omega$ and integrate over a sphere. These two operations give

$$\frac{1}{k_\nu}\frac{dH_\nu}{\rho dx}=-\mathcal{J}_\nu+j_\nu/k_\nu, \tag{2.2}$$

$$\frac{1}{k_\nu}\frac{dK_\nu}{\rho dx}=H_\nu. \tag{2.3}$$

We now proceed to ignore all distinctions of direction except the general distinction of outwards and inwards, and set $\mathcal{J}_\nu(\theta)=(\mathcal{J}_\nu)_1$ when $0<\theta<\pi/2$ and $\mathcal{J}_\nu(\theta)=(\mathcal{J}_\nu)_2$ when $\pi/2<\theta<\pi$. Then

$$\mathcal{J}_\nu=\tfrac{1}{2}\{(\mathcal{J}_\nu)_1+(\mathcal{J}_\nu)_2\},\qquad H_\nu=\tfrac{1}{4}\{(\mathcal{J}_\nu)_1-(\mathcal{J}_\nu)_2\},\qquad K_\nu=\tfrac{1}{3}\mathcal{J}_\nu. \tag{2.4}$$

Now let

$$\int\mathcal{J}_\nu d\nu=\mathcal{J},\qquad\int H_\nu d\nu=H\quad\text{and}\quad\int K_\nu d\nu=K.$$

Further, let

$$\frac{1}{k_2}=\frac{dK}{\rho dx}=\int\frac{1}{k_\nu}\frac{dK_\nu}{\rho dx}d\nu.$$

This defines k_2. Now

$$K=\tfrac{1}{3}\mathcal{J}\quad\text{and}\quad\frac{d\mathcal{J}}{\rho dx}=\frac{d\mathcal{J}}{dT}\frac{dT}{\rho dx},$$

so that we can calculate k_2 from the working formula

$$\frac{1}{k_2}\frac{d\mathcal{J}}{dT}=\int\frac{1}{k_\nu}\frac{d\mathcal{J}_\nu}{dT}d\nu.$$

Writing $\tau_2=\int_x^0 k_2\rho dx$ we now have

$$\frac{dK}{d\tau_2}=H. \tag{2.5}$$

In the individual frequencies H_ν is not constant, partly on account of the formation of the absorption lines and reappearance of the energy in other frequencies, but the integrated flow $H=\int H_\nu d\nu$ is constant. We can integrate (2.5) and using

$$K=\tfrac{1}{3}\mathcal{J}$$

we get

$$\mathcal{J}=3H\tau_2+\text{constant}.$$

From (2.4) we have $\mathcal{J}_\nu=2H_\nu$ at $\tau_2=0$, the surface of the star, where there

is no inflowing radiation and $(\mathcal{J}_\nu)_2 = 0$. Our fundamental equation is accordingly

$$\mathcal{J} = H(2 + 3\tau_2),$$

derived exactly as in *I.C.S.* equation (226.3), except that explicit reference has been made to absorption lines, and that it is now quite clear that k_2, the Rosseland mean value of k_ν, must be used.* The radiation temperature T may be defined by $\mathcal{J} = acT^4$ where a is Stefan's constant, and we get, as in *I.C.S.*, Milne's equation,

$$T^4 = T_0^4(2 + 3\tau_2), \tag{2.6}$$

with $acT_0^4 = 2H$. The effective temperature of the star, T_e, is given by $acT_e^4 = \mathcal{J}_1 = 4H$, so that $T_e^4 = 2T_0^4$.

From here onwards we assume explicitly that k_ν is constant (except where the absorption lines occur) and that its value is k_1. We write $\tau_1 = \int_x^0 k_1 \rho dx$. We assume here explicitly, also, that $j_\nu\, d\nu$, the emission per gramme per sec. between frequencies ν and $\nu + d\nu$, is k_1 times black-body emission at the radiation temperature T of the material, so that

$$j_\nu = ck_1 I(\nu, T),$$

where $I(\nu, T)$ denotes Planck's energy distribution.

The emergent radiation from the star into the solid angle $d\omega$ in the direction making the angle θ with the normal, in the frequency range from ν to $\nu + d\nu$, is

$$d\nu \,.\, \frac{d\omega}{4\pi} \int_\infty^0 j_\nu \rho dx \sec\theta \,.\, e^{-\tau_1 \sec\theta}$$

$$= d\nu \frac{d\omega}{4\pi} \int_0^\infty (j_\nu/k_1) d\tau \sec\theta e^{-\tau_1 \sec\theta}.$$

Denoting the emergent flux by $H(\nu, \theta) \,.\, d\nu \,.\, d\omega/4\pi$ we have

$$H(\nu, \theta) = c \,.\, \int_0^1 I(\nu, T) d\,(\exp(-\tau_1 \sec\theta)). \tag{2.7}$$

This is *I.C.S.* (228·6). The formula gives the following rule: "Divide the range of $e^{-\tau \sec\theta}$ into a large number of equal parts. Calculate the temperatures T_1, T_2, T_3 . . . at the middle of each part. Take a cu. cm. of equilibrium radiation at each of these temperatures; the simple mean gives the constitution of the emergent radiation." We go on to calculate the emergent radiation exactly as in *I.C.S.*, except that the T's in the interior are determined by τ_2 and the extinction of light from the interior is determined by τ_1, where $\tau_2 > \tau_1$.

As a basis of calculation we take Abbot's value $T_e = 5740°$ giving $T_0 = 4830°$. We assume further that the equivalent breadths of the absorption lines in the solar spectrum (weighted according to values of $dI(\nu, T)/dT$, so that the

* This result has been proved by Milne, *Phil. Trans.*, 228, 431, 1929.

maximum weight is at 5500 A.) amount to 11 per cent. of the spectrum. Then, giving the absorption lines infinite coefficients of absorption,

$$\frac{1}{k_2} = 0{\cdot}89\frac{1}{k_1} + 0{\cdot}11\frac{1}{\infty},$$

or

$$k_2 = 1{\cdot}12k_1.$$

We carry out instructions (2.7) in Table I, which corresponds to Table LXIII, p. 327, of *I.C.S.*, the only difference being that the internal temperatures are calculated from τ_2 instead of τ_1. The quantities tabulated under "intensity" are values of $(e^{h\nu/RT} - 1)^{-1}$. We take sec $\theta = 1$, so that the calculations refer to the centre of the disc.

TABLE I

Effect of Absorption Lines on Colour Temperature

$e^{-\tau_1}$	τ_1	τ_2	T	Intensity λ 4157	Intensity λ 6235
0·95	0·0513	0·0575	4931	·00093	·0096
0·85	0·1625	0·1820	5131	·00122	·0115
0·75	0·2877	0·3222	5330	·00157	·0137
0·65	0·4308	0·4825	5534	·00199	·0161
0·55	0·5979	0·6696	5748	·00251	·0188
0·45	0·7985	0·8943	5974	·00316	·0219
0·35	1·0498	1·1758	6228	·00399	·0257
0·25	1·3863	1·5526	6522	·00513	·0305
0·15	1·8971	2·1247	6909	·00691	·0374
0·05	2·9958	3·3552	7570	·01071	·0507
			Mean	·00381	·0236

The colour temperature of the radiation is found from the ratio of the mean intensities, which is 0·0236/·00381 or 6·20, corresponding to 6370°. The corresponding ratio without absorption lines is 6·43, corresponding to 6220°, so that the effect of the 11 per cent. absorption in the lines is to raise the colour temperature by a hundred and fifty degrees.

This method of evaluating the integral is only approximate, as is mentioned by Eddington. The approximate method gives too low a result by about two hundred degrees in the case where there are no absorption lines. According to Milne's more accurate evaluation * of the integral the excess of T_c over T_e at the centre of the disc is about 5·6 per cent. Presumably the same excess of accurate over approximate method applies to the case where we take account of the absorption lines, so that the colour temperature of the centre of the disc given by the model k constant allowing for absorption lines is about $T_c = 6570°$. This is a little below Plaskett's mean observed value. The discrepancy—if it is to be insisted on—means that k increases slightly from violet to red, the hydrogen absorption predominating over the

* *Handbuch der Astrophysik*, 3/1, 153, 1930. An earlier calculation given by the same writer (*M.N.*, 81, 384, 1921) gave the excess as 4·3 per cent., or giving a colour temperature one hundred degrees greater than that given by our approximation.

metallic absorption. It seems absolutely certain that the Pannekoek coefficient, applicable to a purely metallic atmosphere, is quite ruled out by the observations, as is also the hypothesis $j_\nu = \text{constant}$, both of which would give far too low a colour temperature.

The calculations shown in Table I are easy to make, especially if a calculating machine is used, and it has seemed worth while to recalculate the colour temperature for a number of values of sec θ and take a mean colour temperature for the integrated solar disc. The result is $T_c = 6010°$, so that the colour temperature of the whole disc is 360° cooler than that of the centre of the disc. It is still 270° above the effective temperature of the disc. The computations are shown in Table II, the quantities tabulated under mean intensity being mean values of $(\exp h\nu/RT - 1)^{-1}$ for optical depths from 0·05 to 3·00, as used in Table I. Allowing two hundred degrees as before for the imperfections of the method, we may say that the colour temperature of the disc given by the model for k constant is 6200°.

Since Plaskett's observed result for the colour temperature of the centre of the disc is about 6850°, making the above theoretical allowance of 360° for centre over integrated disc, the observed colour temperature of the integrated disc is about 6500°, with a probable error of perhaps two hundred degrees. As has been said above, the excess of observed colour temperature over theoretical colour temperature given by $k = \text{constant}$ indicates that k increases from violet to red.

TABLE II

Colour Temperature from Centre to Limb.

θ	Weight $\sin\theta\cos\theta$	Mean Intensity		T_c
		λ 4157	λ 6235	
5°	·069	·00379	·0235	6370
15°	·250	370	231	6350
25°	·383	351	224	6280
35°	·469	226	212	6210
45°	·500	287	197	6010
55°	·469	244	177	5840
65°	·383	198	155	5620
75°	·250	149	129	5350
85°	·069	093	096	4940
	Weighted mean	·00276	·0190	6010

§ 3. Since the hydrogen absorption from the third state (3S, 3P, 3D) with an excitation potential of twelve volts balances approximately in colour the metallic absorption in the Sun, we find that the same proportion of hydrogen to metals gives an overwhelming preponderance of hydrogen over metallic absorption in the visible region in stellar atmospheres at higher temperatures. In Ao stars, then, unless the hydrogen content is very much less than that of the Sun, the absorption coefficient must resemble, approximately, that due to hydrogen alone, which, as Unsöld has shown, gives a colour temperature greatly in excess of the effective temperature.

There is very little observational evidence for the adoption of any value of the *effective* temperature of the A0 stars, but a method of obtaining effective temperatures from eclipsing variables exists.* Observations of the light curve of an eclipsing variable determine the radii of the two stars in terms of the radius of the relative orbit, and spectroscopic observations of orbital velocity determine the radius of the orbit itself. In a well-observed eclipsing binary, therefore, the radii of the eclipsing stars are known. If, then, there exists a reliable parallax of the system so that the absolute magnitudes of the two stars can be found with any degree of certainty, it is possible to compute an effective temperature for each star. A table of determinations of this sort has been given by Gaposchkin.† This author has, unfortunately, tabulated his final results without stating the parallaxes he uses, and the method is obviously very weak unless the parallax is large. The weakness of the method, unless very great care is taken only to include stars having very good determinations of parallax, is demonstrated by Gaposchkin's results for SX Cas, type A2, 2010°; β Lyr, B5, 5620°, and others. The parallax of β Lyr is 0″·003. These unfortunate results should not shake our confidence in the method if attention is confined to those stars for which larger parallaxes exist, such as β Aur (0″·029), A0, 11,100°; α Cr B, 0″·044, A0, 9730°; Algol, 0″·025, B8, 12,080°—figures which indicate that the effective temperature of the A0 stars must be close to the value 10,000° or 11,000°.

We now come to a very thorny question: What ought the ionisation to be in an atmosphere in which the colour temperature is not equal to the effective temperature? Ionisation—if effected by radiation more than by collisions, which we take to be the case—takes place as the result of absorption at the head of each series in the atomic spectrum. Given an intensity distribution for the radiation it would be possible to calculate the ionisation (under a given electron pressure) if we knew the weight to be assigned to each band, which is proportional to its oscillator strength. Unfortunately, as has been said before, these oscillator strengths are not yet known and we must content ourselves with generalities. Taking a typical metal, calcium, the ionisation potential is about 6 volts, so that the wave-lengths which ionise the atom from the ground state are all shorter than 2030 A. A small change in the colour temperature produces a great effect in this region. For example, suppose for definiteness that the atom is exposed to non-equilibrium radiation with a colour temperature of 6500° and an effective temperature of 5740°; then the radiation of wave-length 2030 A. is $(5740/6500)^4$ × equilibrium radiation at 6500° and is equal in amount to equilibrium radiation at 6200°. There are several important states of *Ca* with excitations between 2 and 4 volts, so that some ionisation will be effected by wave-lengths about 4000 A., for which the radiation is equal to equilibrium radiation at 6000°. Whether the ionisation temperature is nearer 6200° or 6000° will depend on the weighting. It is presumably somewhere between these two limits, and certainly between the colour temperature and the effective temperature.

* Russell, Dugan and Stewart, *Astronomy* (Boston, 1927), pp. 715 *sqq*.

† *Ast. Nach.*, **248**, 213, 1933.

The present writer deduced the ionisation temperature of the Sun * from measurements of the equivalent breadths of the principal lines of Ca^+ 3933 A. and *Ca* 4226 A. A method was put forward for finding n_1/n_0, the ratio of ionised to neutral calcium atoms at the optical depth $\tau = 0{\cdot}3$, from the observed equivalent breadths. In the course of this reduction it is necessary to adopt a value for the ratio of the continuous absorption coefficients at 4226 A. and 3933 A. In the paper referred to the ratio 0·74 was used, following Pannekoek's formula. This gives $n_1/n_0 = 377$ and $T_0 = 5300°$; but if k is constant the value of n_1/n_0 from the observations of equivalent breadth is 280, giving $T_0 = 5145°$. Of course, the fact that k is approximately constant throughout the visible spectrum does not imply a local constancy over short intervals of wave-lengths, and a mixture of hydrogen and metallic absorption which gives equal coefficients in the red and violet has a slightly lower coefficient in the blue-green, so that something of the increase from blue to violet remains. The observed ionisation temperature is reduced slightly by the fact that the electron pressure P will be slightly decreased from the value previously adopted by the distinction between k_1 and k_2. We may say that the observed ionisation temperature is about $T_0 = 5200°$, to be compared with $T_0 = 4830°$ from the effective temperature allowing for absorption lines. These temperatures take on a more familiar aspect if they are all multiplied by $2\frac{1}{4}$, when we get 5740° for the effective temperature and 6180° for the ionisation temperature, which is between the effective temperature and 6500°, the colour temperature of the integrated disc.

Turning to the A-type stars, the hydrogen absorption is predominant and there is a great gap between colour temperature and effective temperature. To get some idea of the ionisation let us take $T_c = 15{,}000°$ and $T_e = 10{,}000°$. The light of wave-length 2030 A. is then about equal in amount to equilibrium radiation at 10,000°, but is only one-fifth of equilibrium radiation at 15,000°: so that the ionisation temperature should be closer to the effective temperature than to the colour temperature. The ionisation temperature depends, presumably, on the excitation potential of the element concerned.

Miss Payne † gives 10,000° for the ionisation temperature of the A0 stars, and Brill ‡ gives 13,500°. In both cases there is a wide departure from the colour temperature of 18,000° given by Greaves, Davidson and Martin.§ We may suppose in this case also that the ionisation temperature lies between the colour temperature and the effective temperature.

It should be said that this paper is the outcome of discussion between Mr W. M. H. Greaves and the writer, whose thanks are further due to Mr Greaves for reading the manuscript and pointing out certain very important considerations.

* *M.N.*, 93, 691, 1933.
† *Stellar Atmospheres*, p. 33, 1925.
‡ *Handbuch d. Astrophysik*, 5/1, 205, 1932.
§ *M.N.*, 94, 488, 1934.

INTERFEROMETRIC DETERMINATIONS OF WAVE-LENGTHS OF SILICON LINES OCCURRING IN B- AND O-TYPE STARS.

C. V. Jackson.

(Communicated by A. Fowler, F.R.S.)

Introductory

The changes which take place in the spectrum of silicon as the conditions of excitation are varied were first investigated over a small range of wave-lengths by Lockyer, who pointed out that the groups of lines which require increasing energy for their production appear in the spectra of stars of successively higher temperatures. By the extension of these observations from the red to the far ultra-violet, Professor A. Fowler identified regular series of lines corresponding with each of the four stages noted by Lockyer and showed that they represented successive stages of ionization of the silicon atom. Thus, while the spectrum of the neutral atom (*Si* I) was characterised by the series constant R, the three other spectra were identified as *Si* II, *Si* III and *Si* IV, from the fact that their series constants were respectively $4R$, $9R$ and $16R$, in accordance with Bohr's theory.* In a later paper, Fowler † gave a more complete description and analysis of the spectrum *Si* I.

The atoms giving rise to the successive spectra, it may be recalled, are usually designated *Si*, Si^{+}, Si^{++} and Si^{+++}, representing respectively the neutral atom, and the singly, doubly and trebly ionized atoms.

The brighter lines of the spectra of singly, doubly and trebly ionized silicon are prominent in the spectra of the stars of the O and B type spectral classes, and since they are usually sharp lines they are very suitable for the determination of radial velocities and have been adopted by the commission on radial velocities of the I.A.U. as primary standards of wave-length ‡ for these stars.

The wave-lengths given in that list were measured against iron standards many years ago, by interpolation from spectrograms of medium dispersion. Since these lines ($\lambda\lambda$ 4088, 4116, 4128, 4130, 4552, 4567 and 4574) are of such great importance astrophysically it was considered advisable to determine their wave-lengths with the highest accuracy attainable by interferometric comparison with the krypton secondary standards. The results are presented below.

* *Phil. Trans.*, A, **225**, 1–48, 1924 : Bakerian Lecture.
† *Proc. Roy. Soc.*, A, **123**, 422, 1929.
‡ *Trans. I.A.U.*, 4, 187, 1932.

Experimental

The spectrum was in all cases produced in vacuum tubes, by passing a condensed discharge through silicon tetrafluoride, as described by Professor Fowler in the Bakerian Lecture of 1924. The gas was excited at low pressures, 0·005 mm. Hg. to 0·05 mm. Hg., as at higher pressures the intensity of the lines of the ionized atoms is relatively reduced. The intensity of the lines of *Si* is increased, as was discovered by Fowler, by using tubes having capillaries of narrow bore, but in this work it was necessary always to use tubes having comparatively wide bores ($1\frac{1}{2}$ mm.) in order to retain the sharpness of the lines.

A condensed discharge, with a spark gap of about half a centimetre, fed by a small Marconi transformer, was used for exciting the tubes. The area of the condenser plates used was either 500 sq. cm. or twice that amount; the larger capacity was used for the plates taken for the wave-lengths for the lines of *Si*.

The degree of excitation on my plates corresponds approximately to that shown in stage No. 2 in Plate I of Professor Fowler's * paper. The *Si* IV lines were therefore rather faint, but by giving exposures of about half an hour they were obtained sufficiently strongly for satisfactory measurement.

Despite the fact that the apparatus had five taps which were freely lubricated with Apiezon grease, not one of the plates showed any carbon impurity lines or bands. This was undoubtedly due to the extremely low vapour pressure of this new vacuum grease.

Etalons of $\frac{1}{4}$ cm. and $\frac{1}{2}$ cm. thickness and films of platinum and of silver on the interferometer plates were used in this investigation. The etalons were carried on a special apparatus which was designed by the author for the purpose of interferometric wave-length determinations and manufactured by Adam Hilger, Ltd.

A Hilger El spectrograph was used for the auxiliary dispersion. For most of the plates the glass train was used, but for about a quarter of the total number the quartz train was used. Ilford Monarch plates were used for nearly all the exposures, on account of their fine grain and great speed in the violet part of the spectrum.

As in my previous work on neon † and krypton ‡ all the plates were taken in the equable temperature room of the Imperial College, so that no trouble was caused by alteration of the optical thickness of the etalons during the course of the comparison. The violet lines of krypton which were adopted by the I.A.U. as secondary standards in 1932 were used as standards for all the exposures.

The procedure was first to photograph the fringes of the krypton lines, then those of the silicon lines, and finally again to photograph the fringes of the krypton lines. Therefore any change in the thickness of the etalon

* A. Fowler, *Phil. Trans.*, *loc. cit.*

† C. V. Jackson, *Proc. Roy. Soc.*, A, **143**, 125, 1933.

‡ *Ibid.*, A, **138**, 148, 1932.

during the course of exposure would have been immediately apparent, when the plate was measured. In none of the plates measured was the change more than two or three thousandths of a fringe.

The Measurements

The method of reduction was the same as that used in my previous work.* Each fringe system of the silicon lines was measured six times and those of the krypton lines three times. The lines of krypton are extremely sharp and therefore very easy to measure. The relative accuracy of the measurements of the krypton lines was such that the mean error of a single plate determination was less than 0·001 A. In all about 40 plates were taken, and of these the 20 best ones were chosen for measurement. The phase correction was applied in the usual way, but it was not necessary to apply any correction to reduce to N.T.P. because of the proximity of the silicon lines to the standards.

The results of the measurements are given below in Table I. The first two columns are self-explanatory, the third column gives the number of plates on which the line was measured, the fourth column gives the mean deviation of a single plate determination from the wave-lengths given in the second column. The greatest deviation was in all cases about twice this amount.

TABLE I.

	I.A. Jackson	*N*	Mean Deviation		Barrell
Si^{+++}	4088·862	6	0·001	A	·863
Si^{+++}	4116·103	6	0·002	B	·104
Si^{+}	4128·051	16	0·001	A	·053
Si^{+}	4130·876	16	0·002	B	·884
Si^{++}	4552·622	16	0·001	A	·611 *
Si^{++}	4567·841	17	0·001	A	·824 *
Si^{++}	4574·758	17	0·001	A	·737 *
Si^{++}	4716·654	8	0·002	B	·658
Si^{+}	5041·035	9	0·001	A	·063
Si^{+}	5056·001	11	0·002	B	·020
Si^{+}	5056·34	1	...	...	·353

* Fowler gives 4552·654, 4567·872 and 4574·777.

The results are thought to be free from systematic errors, because no systematic difference was found between the results derived from plates taken with $\frac{1}{4}$ or $\frac{1}{2}$ cm. etalons or platinum or silver film. It was found that the limiting factor in the accuracy of the measurement of the fringes was the true thickness of the lines, not the instrumental width ; the former was about 0·1 A.

The calculated probable error of all the wave-lengths is under 0·001 A., but the actual accuracy is probably of the order of ± 0·001 A. for the lines marked A and ± 0·002 A. for those marked B in the fifth column.

* C. V. Jackson, *Proc. Roy. Soc.*, A, **130**, 395–410, 1931.

The wave-lengths determined from grating plates by Barrell * in Professor Fowler's laboratory are given in the last column for comparison. These wave-lengths, with the exception of 4552, 4567 and 4574, for which preliminary values were inadvertently substituted, were adopted by Fowler and subsequently by the Radial Velocity Commission of the I.A.U.†

The grating wave-lengths agree well with the new ones for 4088, 4116, 4128 and 4130, the differences being no greater than the experimental error of interpolation by means of a grating of comparatively small dispersion (*ca.* 6 A./mm.). The large discrepancy between the grating wave-lengths of Fowler and those of Barrell for the violet triplet of Si^{++} is almost certainly due to the difficulty of interpolation against the iron standards, which are poor and far too widely separated in this part of the spectrum. The same explanation is applicable to the errors in Barrell's wave-lengths for 5041 and 5056.

Summary

The wave-lengths of the ten brightest lines of silicon occurring in stellar spectra of class B and O have been measured interferometrically with an accuracy of 0·001 A. or 0·002 A., by comparison with the secondary standards of krypton.

In conclusion it is a great pleasure to me to express my gratitude to Professor Fowler for his constant interest and encouragement throughout the course of this work.

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THE SOLAR CHROMOSPHERE (SECOND PAPER).

S. Chandrasekhar, Ph.D.

§ 1. In a previous communication the suggestion was made that the study of the hydrodynamical equilibrium of an atmosphere above a non-uniformly radiating surface (the mean flux from which corresponds to full support by selective radiation-pressure) may give some hints towards our understanding the structure of the chromosphere and its relation to the prominences.‡ In order to examine if this idea would work, the mathe-

* *M.N.*, **83**, 322, 1923.

† *Trans. I.A.U.*, **4**, 187, 1932.

‡ *M.N.*, **94**, 14, 1933. Referred to as I. See also *Observatory*, **57**, 65, 1934.

My attention has recently been drawn to an interesting paper by E. J. Perepelkin, "Über die Struktur der Sonnen Chromosphäre" (*Z. f. Astrophysik*, **6**, 245, 1933), where the consequences of the *assumption* that the solar chromosphere is made up of small filaments with heights distributed exponentially are examined. Though the fundamental standpoint taken in I is essentially different from that taken by Perepelkin in his analysis, yet both the methods of approach attempt a generalization of our existing *views* on the chromospheric equilibrium in the same direction.

matical solution for a specially idealized model was worked out and the results confirmed the belief that this possibility would have to be carefully examined before one abandons altogether the theory of "support" by selective radiation-pressure. Further, the very recent analysis of the dark H_α markings on the surface of the Sun which has since been published by Newton * has proved convincingly at any rate the logical utility of the theory. It is of some importance, then, to obtain the solutions for the mathematical problem under less restrictive conditions, and the present paper is an attempt to generalize the analysis contained in I and also to examine carefully the physical premises that underlie the theory.

I. *The Effect of Darkening*

§ 2. In I we assumed that the intensity of the radiation at any point (x, y, z) of the radiating surface was locally isotropic in the hemisphere. Actually, however, we know that under the solar conditions the intensity of radiation at any point (x, y) and in a direction making an angle θ with the normal at (x, y) can be expressed as

$$I(x, y, \theta) = I(x, y)(1 - u + u \cos \theta), \tag{1}$$

where now $I(x, y)$ is independent of the direction and u is the so-called coefficient of darkening. The theoretical law of darkening, confirmed by observation, gives for u the value $3/5$. We shall accordingly use this value of u in our numerical work.

As before we assume that

$$I(x, y) = I_0 + I_1(x, y), \tag{1'}$$

and further that the atom would be in equilibrium when the flux of radiation is that corresponding to I_0.

Then instead of I, (2), we have

$$\tfrac{1}{4}B_{1\to 2}(1 - \tfrac{1}{3}u)I_0 = mg. \tag{2}$$

Confining ourselves to the case

$$I_1(x, y) = I_1 \sin \frac{2\pi}{\lambda}x, \tag{3}$$

we find after some algebra that our fundamental system of dynamical equations of motion are

$$\frac{d^2\xi}{dt^2} = a \cos \xi[q_1\zeta K_0(\zeta) + q_2\zeta e^{-\zeta}], \tag{4}$$

$$\frac{d^2\zeta}{dt^2} = a \sin \xi[q_1\zeta K_1(\zeta) + q_2(\zeta + 1)e^{-\zeta}], \tag{4'}$$

where

$$q_1 = \frac{3(1-u)}{3-u}; \qquad q_2 = \frac{2u}{3-u}, \tag{5}$$

* H. W. Newton, *M.N.*, 94, 472, 1934. See also the discussion of this paper reported in the *Observatory*, 57, 142, 1934. I hope to consider in detail a theoretical explanation of Mr. Newton's interesting observations on a future occasion.

and the rest of the symbols having the same meaning as in I. For case $u = 3/5$, we notice that

$$q_1 = q_2 = 1/2. \tag{6}$$

§ 3. *The Solutions of the Equations Motion.*—As in I, § 8, we find that the equations of motion (4) possess the following integral :—

$$\frac{d\xi}{dt} \cdot \frac{d\zeta}{dt} + a \cos \xi [q_1 \zeta K_1(\zeta) + q_2(\zeta + 1)e^{-\zeta}] = \text{constant}. \tag{7}$$

We can prove that for periodic solutions *permanently in the chromosphere*

$$\frac{d\xi}{dt} \cdot \frac{d\zeta}{dt} = -a \cos \xi [q_1 \zeta K_1(\zeta) + q_2(\zeta + 1)e^{-\zeta}]. \tag{8}$$

From (8) and the equations (4) we derive

$$\frac{d\xi}{dt} = \alpha [q_1 \zeta K_1(\zeta) + q_2(\zeta + 1)e^{-\zeta}], \tag{9}$$

$$\frac{d\zeta}{dt} = -\beta \cos \xi, \tag{10}$$

where as before

$$\alpha\beta = a. \tag{11}$$

From (9) and (10) we derive for the equation of the trajectory

$$\sin \xi = \frac{\alpha}{\beta} [q_1 Q(\zeta_0, \zeta) + q_2 P(\zeta_0, \zeta)], \tag{12}$$

where

$$Q(\zeta_1, \zeta_2) = \int_{\zeta_2}^{\zeta_1} \zeta K_1(\zeta) d\zeta, \tag{13}$$

and

$$P(\zeta_1, \zeta_2) = \left[-(\zeta + 2)e^{-\zeta} \right]_{\zeta_2}^{\zeta_1}. \tag{14}$$

If we now define

$$R(\zeta_1, \zeta_2) = q_1 Q(\zeta_1, \zeta_2) + q_2 P(\zeta_1, \zeta_2), \tag{15}$$

we can then rewrite (12) in the form

$$\sin \xi = \frac{\alpha}{\beta} R(\zeta_0, \zeta). \tag{16}$$

From (13) and (14) we have

$$R(\infty, 0) = \tfrac{1}{2}(\pi q_1 + 4q_2). \tag{16'}$$

For the atoms that are in the chromosphere we should consider only such trajectories that they do not either intersect the ξ-axis or go off to infinity. This discussion can be carried out exactly as in I, § 11, and the following table now replaces Table I of I. (The calculations have been done for the case $u = 3/5$, *i.e.* for $q_1 = q_2 = 1/2$.)

TABLE I *

ζ_0	$\alpha_{\min} \cdot a^{-1/2}$	$\beta_{\max} \cdot a^{-1/2}$	Minimum Horizontal Velocities
0	∞	0	∞
	0·7484	1·3362	0·7484
0·5	1·4521	0·6887	1·2619
	0·8733	1·1451	0·7589
1·0	1·0790	0·9268	0·7217
	1·0389	0·9625	0·6949
1·5	1·2510	0·7993	0·6092
2·0	1·5195	0·6581	0·5210
2·5	1·8575	0·5384	0·4384
3·0	2·2822	0·4382	0·3647
3·5	2·8155	0·3552	0·3009
4·0	3·4849	0·2870	0·2466
4·5	4·3258	0·2312	0·2011
∞	∞	0	0

In the figure the trajectories calculated on the basis of the new equations are illustrated. The general forms are the same as before.

As in I, § 12, we can calculate the limiting velocities attained by the atoms starting from the point $(0, \lambda/4)$ with zero horizontal velocity.† One finds that

$$\dot{\zeta}_\infty^2 = \dot{\zeta}_0^2 + a(\pi q_1 + 4q_2), \tag{17}$$

or if $\dot{\zeta}_0 = 0$ we have (*cf.* I (48))

$$\dot{\zeta}_\infty = \sqrt{rg \cdot \frac{\lambda}{2}\left(q_1 + \frac{4}{\pi}q_2\right)} \to 32 \text{ km./sec.}^{-1}. \tag{18}$$

§ 4. *The Density Distribution in the Chromosphere.*—In determining the density distribution we impose the condition as in I, § 13, that $\dot{\xi}$ and $\dot{\zeta}$ are to be unique functions in the whole ξ-ζ plane. This means that we will have to consider only such systems of trajectories which are characterized by a certain assigned value for α/β.

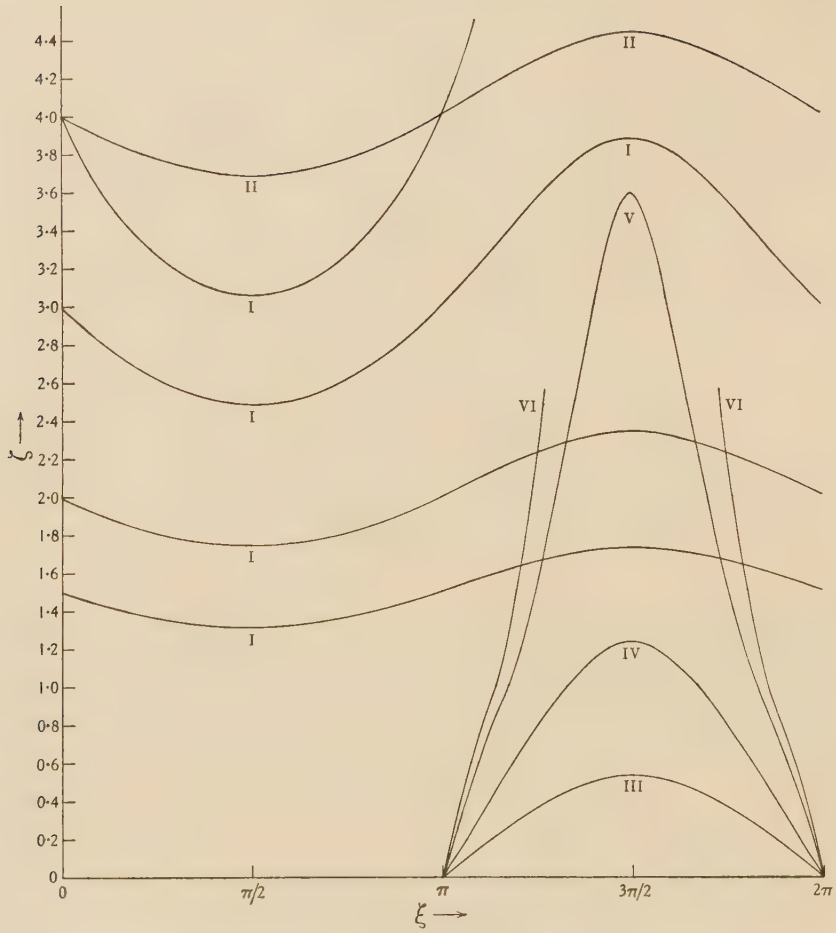
From the equation of continuity we now find that

$$\rho = \Phi\left\{\sin\xi + \frac{\alpha}{\beta}R(\zeta, 0)\right\}, \tag{19}$$

where $\Phi(x)$ is used to denote an arbitrary function of the argument x.

* For $\zeta_0 = 0$, 0·5, 1·0 a pair of values for $\alpha_{\min} \cdot a^{-1/2}$, etc., are given. The first values correspond to the figures given in I, and the second values provide the maximum *upward* vertical velocity ($= \beta_{\max} \cdot a^{-1/2}$), etc., for the atoms starting at (ζ_0, π) not to go off to infinity. A study of these trajectories becomes important in view of Newton's work (see the trajectories in the second half of the figure).

† The possibility of the emission of such moderately high-speed particles, which was predicted in I, has since been observationally discovered by Mr. Newton (*loc. cit.*).



Trajectories of the Chromospheric Atoms.
I→$\alpha/\beta=10$; II→$\alpha/\beta=40$; III→$\alpha/\beta=2$;
IV→$\alpha/\beta=1$; V→$\alpha/\beta=0{\cdot}6$; VI→$\alpha/\beta=0{\cdot}5$.

Corresponding to the approximation in I, § 14, we now deduce that to a first approximation the density distribution is specified by the relation

$$\rho=\rho_0\cdot\frac{2}{\pi q_1+4q_2}R(\infty,\zeta). \tag{20}$$

(20) now predicts for the march of the density-gradient the law

$$\frac{\partial\rho}{\partial\zeta}=-\rho_0\cdot\frac{2}{\pi q_1+4q_2}\{q_1\zeta\kappa_1(\zeta)+q_2(\zeta+1)e^{-\zeta}\}. \tag{21}$$

In Table II we have a comparison of the above law of the density gradient with the purely exponential law. The general agreement is the same as before.

TABLE II

ζ	$-\frac{1}{\rho_0}\frac{\partial\rho}{\partial\zeta}$	$e^{-\zeta}$	ζ	$-\frac{1}{\rho_0}\frac{\partial\rho}{\partial\zeta}$	$e^{-\zeta}$
0	0·5601	1·0000	2·5	0·1322	0·0821
0·5	0·4867	0·6065	3·0	0·0895	0·0498
1·0	0·3746	0·3679	3·5	0·0599	0·0302
1·5	0·2727	0·2231	4·0	0·0396	0·0183
2·0	0·1920	0·1353	4·5	0·0260	0·0111

For the total mass M supported in a unit column of the chromosphere one finds

$$M=\lambda\rho_0\cdot\frac{2q_1+3q_2}{\pi(\pi q_1+4q_2)}, \tag{22}$$

or if N is the number of atoms (in that stage of ionization and excitation in which it is directly supported by radiation-pressure) in the unit column and n_0 the number per unit volume at the base, then

$$N=n_0\cdot\frac{(2q_1+3q_2)\lambda}{\pi(\pi q_1+4q_2)}, \tag{23}$$

$$=0\cdot2229n_0\lambda, \text{ if } q_1=q_2=\tfrac{1}{2}. \tag{23'}$$

(23′) is to be compared with our previous result I, (58).

As in I, we now find for the mean value of ξ the expression

$$\bar{\xi}=\frac{(\pi q_1+4q_2)^2}{8(2q_1+3q_2)}\alpha, \tag{24}$$

$$=4\cdot033\sqrt{\alpha} \qquad (q_1=q_2=2^{-1};\ \alpha=40^{1/2}). \tag{24'}$$

§ 5. A second approximation to the density distribution can now be determined as in I, § 15.

Assuming that at the base of the chromosphere the variation of density is given by

$$\rho_{\text{base}}=\rho_0(1-\epsilon\sin\xi), \tag{25}$$

we have by (19)

$$\rho=\rho_0\{(1-\epsilon\sin\xi)-\epsilon\alpha\beta^{-1}R(\zeta,0)\}. \tag{26}$$

When

$$\epsilon(R(\infty,0)\alpha\beta^{-1}-1)\geqslant 1, \tag{27}$$

an atmosphere governed by the law (26) has a sharp boundary ζ_B defined by

$$R(\zeta_B,0)=\beta\alpha^{-1}(\epsilon^{-1}-\sin\xi). \tag{28}$$

But an atmosphere with an assigned value for α/β has a *natural* boundary

specified by the equations

$$\alpha\beta^{-1}R(\infty, \zeta_N) = 1, \tag{29}$$

$$\alpha\beta^{-1}R(\zeta_N, \zeta) = \sin\xi. \tag{29'}$$

The condition that the natural boundary should coincide with the actual boundary (28) will determine ϵ. One finds, in fact, that (27) should now be an *equality*. Thus in our second approximation we have

$$\frac{\partial\rho}{\partial\zeta} = -\rho_0 \frac{1}{R(\infty, 0)}\left(1 - \frac{\alpha}{\beta R(\infty, 0)}\right)^{-1}\{q_1\zeta K_1(\zeta) + q_2(\zeta+1)e^{-\zeta}\}, \tag{30}$$

which reduces to our earlier formula (21) for large α/β—as it should.

II. *A First Discussion of the Two Dimensional Problems*

§ 6. In the previous sections we have generalized the treatment given of the chromospheric problem in I to take into account the effect of darkening and found that this latter effect introduces no essential modification in the general theory. But it still remains to examine if the characteristics of the chromospheric equilibrium which we have derived on the basis of a simplified model with a one-dimensional variation of flux would also be retained when the problem is treated in a sufficiently general way. It is clear, for instance, that the real chromosphere would correspond more nearly to the ideal case of a two-dimensional variation of flux, *i.e.* we should more accurately assume for the intensity of radiation at a point (x, y) on the radiating surface and at an angle θ to the normal at (x, y) the expression

$$I = \left\{I_0 + I_1 \sin\frac{2\pi}{\lambda_x}x \sin\frac{2\pi}{\lambda_y}y\right\}(1 - u + u\cos\theta). \tag{31}$$

The equations of motion can be worked out, and one finds after some rather lengthy algebra that

$$\frac{d^2\xi}{dt^2} = a_\xi \cos\xi \sin\eta\{q_1 Z K_0(Z) + q_2 Z e^{-Z}], \tag{32}$$

$$\frac{d^2\eta}{dt^2} = a_\eta \sin\xi \cos\eta[q_1 Z K_0(Z) + q_2 Z e^{-Z}], \tag{33}$$

$$\frac{d^2Z}{dt^2} = (a_\xi + a_\eta)\sin\xi \sin\eta[q_1 Z K_1(Z) + q_2(Z+1)e^{-Z}], \tag{34}$$

where

$$\xi = \frac{2\pi}{\lambda_x}x; \qquad \eta = \frac{2\pi}{\lambda_y}y; \qquad Z = \left\{\left(\frac{2\pi}{\lambda_x}\right)^2 + \left(\frac{2\pi}{\lambda_y}\right)^2\right\}^{1/2} z, \tag{35}$$

and

$$a_\xi = \frac{I_1}{I_0}\left(\frac{2\pi}{\lambda_x}\right)^2\left\{\left(\frac{2\pi}{\lambda_x}\right)^2 + \left(\frac{2\pi}{\lambda_y}\right)^2\right\}^{-1/2} g; \qquad a_\eta = \frac{I_1}{I_0}\left(\frac{2\pi}{\lambda_y}\right)^2\left\{\left(\frac{2\pi}{\lambda_x}\right)^2 + \left(\frac{2\pi}{\lambda_y}\right)^2\right\}^{-1/2} g. \tag{36}$$

In practice we can assume that $\lambda_x = \lambda_y$. For this special case we can conveniently rewrite the equations (32)–(34) as follows: Introduce the new variables

$$X = \xi + \eta - \pi/2, \tag{37}$$

$$Y = \pi/2 - (\xi - \eta). \tag{37'}$$

This transformation is merely equivalent to a rotation of (ξ, η) axis by an angle of 45° in the (ξ, η) plane, decreasing our unit of length $(=\lambda/2\pi)$ by a factor $\sqrt{2}$ and then finally translating our new system of axes in the (ξ, η) plane to another parallel system. Our equations of motion are

$$\left.\begin{aligned} \ddot{X} &= -A \cos X f'(Z) \\ \ddot{Y} &= -A \cos Y f'(Z) \\ \ddot{Z} &= A(\sin X + \sin Y) f(Z) \end{aligned}\right\}, \tag{38}$$

where

$$f(Z) = q_1 Z K_1(Z) + q_2(Z+1)e^{-Z}, \tag{39}$$

and

$$A = \tfrac{1}{2}\frac{I_1}{I_0}\frac{2\pi\sqrt{2}}{\lambda} g. \tag{40}$$

It is now clear that we are now measuring distances along the three axes on the same scale, and that, further, our fundamental unit of length is $\{\lambda/(2^{\frac{1}{2}})2\pi\}$. We easily verify that the equations (38) are consistent with our previous system of equations (4).

A particular solution of (38) is obviously $X = Y$, in which case our equations of motion are

$$\left.\begin{aligned} \ddot{X} &= -A \cos X f'(Z) \\ \ddot{Z} &= 2A \sin X f(Z) \end{aligned}\right\}, \tag{41}$$

which formally differs from our previous system (4) only by having a factor 2 in the second equation. I have not yet found it possible to obtain the solutions of (38) or even of the pair (41) by purely analytical methods, but have verified by "trial and error" that periodic solutions do exist. It is also clear that the trajectories on this model would now be of a less fluctuating character, and that, further, the order of magnitude of the velocities will not be very different. Thus an atom starting at the point $X = Y = \pi/2$ with zero horizontal velocity is continually accelerated up the vertical, and one easily finds that along these special trajectories

$$\dot{Z}^2 = \dot{Z}_0^2 + 4A\int_0^Z f(Z)dZ, \tag{42}$$

or

$$\dot{Z}_\infty^2 = \dot{Z}_0^2 + 2A(\pi q_1 + 4q_2), \tag{43}$$

which is exactly of the same form as (17). The limiting velocities of particles with zero initial velocities is for a given λ, $2^{\frac{1}{2}}$ times that predicted on the basis of our one dimensional model. This factor, of course, is not of much significance.

We may also expect from analogy with the one dimensional case that the

condition of the uniqueness of the components $\dot{X}$, $\dot{Y}$, $\dot{Z}$ in the whole of the (X, Y, Z) space would introduce certain restrictions and that an atmosphere set up consistent with these restrictions will have a "natural" boundary. The density distribution should then be so determined that it firstly satisfies the hydrodynamical equation of continuity, and secondly vanishes on the natural boundary. Now since the natural boundary would be specified by a system of trajectories, we can always find density distributions satisfying the two conditions stated above, for according to a general theorem in hydrodynamics *we can always find density distributions consistent with the equation of continuity and vanishing on an arbitrarily specified system of trajectories.*

[As the above theorem does not appear to have been explicitly stated in any book on hydrodynamics the following proof may be noted here :—

Let the trajectories be known in the form

$$\left.\begin{aligned}\phi(x, y, z) &= a\\ \psi(x, y, z) &= b\end{aligned}\right\}. \tag{44}$$

Then the tangent to the curve is

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz &= 0\\ \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz &= 0\end{aligned}\right\}. \tag{45}$$

From (45) we deduce that

$$\frac{u}{\dfrac{\partial(\phi, \psi)}{\partial(y, z)}} = \frac{v}{\dfrac{\partial(\phi, \psi)}{\partial(z, x)}} = \frac{w}{\dfrac{\partial(\phi, \psi)}{\partial(x, y)}} = \Omega \text{ (say)}, \tag{46}$$

where u, v, w are the components of the velocity. Since u, v, w are known, Ω is a known function of x, y, z. Now the equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{47}$$

We have now from (46) and (47) that

$$\frac{\partial(\phi, \psi)}{\partial(y, z)}\frac{\partial}{\partial x}(\rho\Omega) + \frac{\partial(\phi, \psi)}{\partial(z, x)}\frac{\partial}{\partial y}(\rho\Omega) + \frac{\partial(\phi, \psi)}{\partial(x, y)}\frac{\partial}{\partial z}(\rho\Omega) = 0, \tag{48}$$

or

$$\frac{\partial(\rho\Omega, \phi, \psi)}{\partial(x, y, z)} = 0. \tag{49}$$

Hence

$$\rho\Omega = F(\phi, \psi), \tag{50}$$

where F denotes an arbitrary function of the arguments. From (50) the theorem stated follows at once.]

A detailed treatment of the system of equations (38) will be undertaken in a future communication, but meantime we may conclude (at least provisionally) that the general characteristics of the chromospheric equilibrium we have derived from our analysis of the one dimensional case will not be essentially modified.

III. *A General Discussion of the Physical Premises Underlying the Theory*

§ 7. *Absorption in the Chromosphere.*—The theory as outlined in I, § 4, and elaborated in the previous sections, obviously applies to the calcium chromosphere, for the underlying hypothesis that the atom is fully supported by radiation-pressure against gravity by the flux of radiation corresponding to the mean intensity (I_0) is true, as was first shown by Milne. Further, in the case of the calcium chromosphere, since the main support is derived by absorption due to a principal line in the intense part of the spectrum, we do not meet here the difficulties associated with the hydrogen chromosphere. Also in our treatment we do not encounter the difficulties connected with ionization in as much as we cannot in any case make a straightforward application of the ionization formulæ. A complete description of the chromospheric equilibrium has been obtained on the present theory without at any stage introducing the conception of temperature.

There is, however, one point which will have to be considered, namely, the reduction in the outward flux of radiation by the absorption in the chromosphere. In Milne's original hydrostatic theory of the chromosphere this reduction in the outward flux of radiation was essential to the problem. In fact, in his theory the base of the chromosphere was placed at such a level where the outward flux of radiation was equal to the photospheric flux. In defining the base of the chromosphere in this way we are clearly including the reversing layers in the more conventional chromosphere, but this is not a bad approximation, since the extension of the reversing layers must be small compared with the extension of the chromosphere and in so far as the above way of defining the base of the chromosphere was used only to determine a certain constant of integration. But it is also clear that in so doing we must be encountering in some form or other the difficulties connected with the residual intensities in absorption lines. Indeed, the difficulties in the hydrostatic theory pointed out by Menzel * are in a sense only formal difficulties arising from the fact that Milne's theory of the chromosphere is at the same time a theory of the *contour* of the chromospheric absorption lines as well. We avoid these formal difficulties in our hydrodynamical theory since we have assumed that the absorption line has *already* been formed in the reversing layers, and further, the radiating surface which the theory postulates is placed above the reversing layers.† Now the number of atoms in a unit column of the reversing layers is of the order of 10^{18} per cm.^{2} (of course, we encounter here the serious difficulty in the theory of the contours of absorption lines that the number of atoms required to explain the "widths" of the absorption lines would, in fact, completely blacken the centre; but in a theory of the *chromosphere* we are not concerned with these difficulties). However, it is clear that a further addition of 10^{14} atoms ‡ above the reversing layers would hardly affect the outward flux of radiation. Actually if we calculated the reduction in the

* D. H. Menzel, *M.N.*, **91**, 628, 1931. † Compare I, footnote on p. 16.

‡ We *know* from observations that this is the order of magnitude of the number of atoms in a column of the chromosphere.

outward flux at the centre of the line (by a straightforward application of Schuster's formula) due to these $\sim 10^{14}$ atoms, we would find that the reduction—in theory—is quite appreciable; but clearly we cannot assume that this *calculated* reduction in the outward flux is "real" when we remember the corresponding situation in the theory of the ordinary absorption lines. We are therefore, when studying the chromospheric equilibrium, justified in neglecting the reduction in the outward flux by absorption in the chromosphere.

§ 8. In connection with the remarks in § 7 we can clear up a formal paradox which one can deduce from our result that "the number of atoms N in a unit column of the chromosphere is equal to the number of atoms in a unit column of about a fifth (or more exactly 0·22) of the wave-length λ filled uniformly at the concentration occurring at the base of the chromosphere." * Now if one made λ tend to infinity then the number of atoms N would be infinite, and this contradicts the fact that when $\lambda \to \infty$ we should expect our theory to pass over into Milne's hydrostatic theory. The solution to this paradox lies in this, that for the hydrodynamical theory to be valid it is essential that the reduction in the outward flux of radiation by absorption in the chromosphere could be neglected, or, to put it more precisely, it is an essential physical assumption in our theory that the number of atoms in a column of about a fifth of the *assumed* wave-length λ at the density occurring at the base of the chromosphere should be very small compared with the "number of atoms above the photosphere" that exists in the reversing layers and as calculated by the method of Stewart and Unsöld. The consistency of the hydrodynamical solution depends on the fact that under the *observed* conditions existing in the solar atmosphere the assumption is in fact justifiable.

§ 9. *The Chromospheric Equilibrium for the Case where the Atom has no Principal Lines in the Intense Part of the Spectrum.*—The hydrogen chromosphere belongs to this class. At first sight it would appear that the hydrodynamical theory would not apply to this case, but it is clear that the whole theory would formally be applicable to any species of atoms, provided only the atom in that specified stage of ionization and excitation can be fully supported by the mean selective radiation-pressure, *i.e.* if κ is the appropriate absorption coefficient of the atom (in the particular stage of ionization and excitation) we should have

$$\tfrac{1}{4}\kappa(1-\tfrac{1}{3}u)I_0 = mg. \qquad (51)$$

The theory would be applicable to this species of atom only, and it would not be logical to conclude that a "proportionate" number of atoms in other stages of ionization and excitation would be found in the high-level chromosphere. This is further understandable from our result that $N = 0{\cdot}223 n_0 \lambda$. The effect of the macroscopic accelerations is to transfer atoms (which can be fully supported by radiation-pressure) from the top of the reversing layers into the high-level chromosphere. The atoms would initially be shot up from above the hotter regions, and these atoms would then "feel

* *Cf.* I, p. 28.

their way" towards a steady state consistent with the equation of continuity. In the case of atoms which like Ca^+ have principal lines in the intense part of the spectrum there would be no difficulty in such a steady state being set up. The problem, however, is not quite so easy with the hydrogen atoms, but the theory definitely gives a clue as to how high-level chromospheres of such atoms could be set up. It is further clear that such chromospheres, unlike the calcium chromosphere, should perpetually be in a state of being formed, and this leads us to expect that the hydrogen chromosphere should have much more of a "local" character than the calcium chromosphere. A rigorous investigation of the equilibrium of the hydrogen chromosphere on these lines should be possible, but the point which has to be emphasised in this connection is that the theory which we have developed, though, strictly speaking, applicable only to the calcium chromosphere, is yet sufficiently general that an extension of the theory to the other cases should not be altogether impossible.

Summary.—The analysis contained in a previous paper on "The Solar Chromosphere" is generalized to take into account the effect of darkening. A first discussion of the equilibrium above a radiating surface with a two-dimensional variation of intensity is also made, and reasons advanced for believing that the characteristics of the structure of the chromosphere derived on the basis of a model with a one-dimensional variation of flux will be true even if the problem were solved with sufficient generality. Finally, the physical premises underlying the theory are carefully examined.

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NON-STATIC HYDROGEN CHROMOSPHERES AND THE PROBLEM OF Be STARS.

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Summary and Introduction

The possible mechanisms producing dissipation of stellar atmospheres are considered (§ 1). It is shown (§ 2) that in the atmospheres of some luminous B stars radiative acceleration overbalances gravity for hydrogen atoms, thus leading to various dissipation phenomena. The radiation field in the nebular layers of a schematic hydrogen extended chromosphere is analysed (§ 4). The mechanical state of a hydrogen non-static chromosphere is then investigated and the necessary conditions of steady outflow derived (§ 5). This leads to the theoretical possibility of various kinds of non-static chromospheres. In § 6 this theory is applied to explain the P Cygni characteristics, and a model of the P Cygni non-static chromosphere is given. The question of the acquisition of initial velocity is briefly discussed in § 7, which contains also the estimate of the limiting velocity attained by "the forerunner" of the freshly formed moving strata. The insufficient luminosity of Bne stars and the lack of considerable asymmetry—"absorption wings-broad emission"—

prevent the straightforward application of the theory to these stars (§ 8). It appears that the stellar rotation, working against gravity, facilitates radiative dissipation, which, however, cannot be of a steady character. The problem of Bse stars and the luminosity sequence of Be stars is considered in § 9. The appendix contains an additional analysis of the transmission of radiation through moving media (the constancy of the net flux and the equation of radiative transfer).

Considerable progress in the interpretation of the emission spectra in celestial objects has been made during the last six years. Zanstra's theory of the nebular emission, Bowen's analysis of the nebular spectra, Edlen, Payne and Menzel's identifications of lines in Novæ and Wolf-Rayet stars, Beals and Menzel's hypothesis on the nature of Wolf-Rayet stars have all contributed to the elucidation of the problem. Equal progress has been made in investigating Be stars. The observational material collected at Harvard, Michigan, Norman Lockyer and Mount Wilson Observatories constitutes a reliable background for theoretical investigations. Further, O. Struve * has shown that the relative intensities of Balmer emission lines in Be spectra are essentially the same as those in nebular spectra. This discovery is of a crucial character, and proves that hydrogen emission in Be spectra is produced by recombinations as in the nebulæ, stimulated by neighbouring stars. Working on Struve's ideas and using Zanstra's method, O. Mohler † has determined the effective temperatures of Be stars. All these results remove any reasonable doubt that the Be emissions originate in nebular shells enveloping these stars, thus making unlikely Eddington's ‡ suggestion to the effect that its source can be looked for in static stellar atmospheres. From a purely physical point of view Be emission seems to be explained unambiguously, and the whole question reduces to the general nebular problem.

However important this progress is, one should not forget that this is only the first step in the solution of the general problem of Be stars. This problem cannot be considered as solved until we explain the origin of these nebular envelopes surrounding Be stars, interpreting at the same time the contours of the emission lines and their variations. An ideal theory should, moreover, account for the affiliation of Be stars and their general statistical properties. It seems obvious that the Be nebulæ consist of atoms in some way or other ejected from central stars. This ejection must have been powerful enough in order to make nebular envelopes observable and to a certain measure steady. In other words, there must be some epochs in the evolution of the hot stars when their atmospheres become unstable and some cause starts to work dissipating them partially into interstellar space. The nature of this dissipation constitutes the main question in the Be stars problem in its modern form.

It is the purpose of this paper to work out a definite theory of these phenomena. This theory seems to account for the observed phenomena in Be stars.

* Series of papers in *Ap. J.* especially ; Struve and Swings in *Phys. Review*, **38**, 1195, 1931 ; see also Miss Payne's paper in *M.N.*, **92**, 368, 1932.

† *Mich. Publ.*, **5**, 43, 1933. ‡ *I.C.S.*, p. 343, 1926.

§ 1. *Dissipation Phenomena in the Atmospheres of Hot Stars*

At the present time we may indicate three theoretically possible mechanisms working against gravity and producing partial or total dissipation of stellar atmospheres: (1) thermal escape; (2) rotation; and (3) selective radiation pressure. All these causes will be analysed separately under corresponding headings.

(1) *Thermal Dissipation.*—This mechanism is included here only for the sake of completeness. Even in the hottest supergiants thermal velocities are too small to produce the loss of matter of any astrophysical importance. On the other hand, the growth of a macroscopic charge resulting from the escape of the electrons is soon checked by the reverse process. For the details see Milne's well-known memoir.*

(2) *Rotational Instability.*—If the star possess a large angular momentum it may become unstable, and under certain conditions its excessive angular momentum may be relieved by the equatorial ejection of matter instead of breaking up into two parts after passing through pear-shaped figures of equilibrium. Arguing from the ideas of Laplace and Roche, one may expect that such an equatorial instability would result in the formation of a ring in the equatorial plane endowed with rapid rotation. Such a nebular ring would produce emission contours depending upon the inclination of the line of sight to the rotational equator, minimum intensity always corresponding to zero velocity. These ideas represent the starting-point for the only existing hypothesis on the origin of Be emission as recently advanced by O. Struve.†

Struve suggests that extremely flat and broad absorption lines underlying nebular emission in Bne stars are due to rapid axial rotation of central stars. On the other hand, the widths of the emission lines in these stars defined as the distance between the steepest outer gradients of emission show, according to R. Curtiss, a pronounced proportionality to wave-length, thus indicating the effect of rotation in the nebular ring. The widths of the single lines as well as the distance between the components of the double lines are not the same for all stars, and depend, as Struve suggests, upon the inclination of the axis of rotation to the line of sight. There exists also a well-marked correlation between the widths of emission and the degree of rotation, as defined by the widths of stellar lines.

(3) *Radiative Dissipation.*—The expulsion of atoms from the hot stars under radiation pressure has been investigated by several authors. Milne ‡ has considered the expulsion of a relatively small number of atoms which do not give an appreciable absorption line and move into the violet wing of the stellar absorption line and even climb right out into the continuous spectrum, thus acquiring high limiting velocities. He, however, does not investigate the problem of the way appreciable initial velocities are acquired. The possibility of radiation pressure counterbalancing gravity at a stellar surface for various atoms has been considered in details by M. C. Johnson.§ Johnson finds that for temperatures $\leqslant 30{,}000^\circ$ and $p_e = 10^{-4}$

* *Trans. Camb. Phil. Soc.*, **22**, 483, 1923. † *Op. cit.*

‡ *M.N.*, **86**, 459, 1926. § *Ibid.*, **85**, 813, 1925, and **86**, 300, 1926.

atm. radiative acceleration for the ions is in no case greater than gravitational (for H it is 10^{-3} times smaller), except for *He* at 30,000°. This result does not give us any hope of explaining the origin of Be nebular envelopes by radiative dissipation. On the other hand, P Cygni spectra clearly manifest the ejection of atoms from these stars, which in no way can be explained by rotational instability. One can hardly escape the conclusion that only by the action of radiation pressure can we explain such a steady shedding off of the hydrogen atoms from P Cygni stars. Though this argument is clearly of the *faûte de mieux* character, it is important to reconsider this question with respect to the most important constituent of Be nebulæ, namely, hydrogen.

§ 2. *The Velocities of Hydrogen Atoms under Radiation Pressure in a Stellar Atmosphere*

Calculating radiative accelerations for various atoms and lines, Johnson has used throughout a mass absorption coefficient of 10^8, which gives an atomic absorption coefficient for hydrogen of $1{\cdot}7 \times 10^{-16}$. This value is certainly too small if one deals with the most prominent lines of the Lyman series. For such computations the "Doppler absorption coefficient" should obviously be used, which if averaged over the Doppler width is $1{\cdot}3 \times 10^{-2}\frac{cf}{\nu_0 v_t}$, where f is "the number of dispersion electrons per atom" and v_t the mean thermal velocity.* For L_α and $T = 20{,}000°$ this absorption coefficient comes out to be $3{\cdot}6 \times 10^{-14}$, *i.e.* 210 times larger than that used by Johnson. This underestimate is only partly compensated by his using a line width of $2{\cdot}10^{-8}$ cm. instead of the Doppler width for L_α ($1{\cdot}2{\cdot}10^{-9}$ cm.). On the other hand, calculating the intensity available beyond the Lyman limit, Johnson uses the same value $\Delta\lambda = 2{\cdot}10^{-8}$ cm., thus substituting for the large range in ν beyond the Lyman limit a very minute quantity. This makes us suspect whether Johnson's values for radiative pressure on hydrogen atoms in stellar atmospheres are not too low.

Consider a representative Be atmosphere for which we may adopt an effective temperature between 20,000° and 15,000°. The acceleration gained by the hydrogen atom in the forward direction from ionizing frequencies beyond the Lyman limit is

$$j_{\text{rad.}} = \frac{(1-x) f_1 r \pi}{m_H c} \int_{\nu_1}^{\infty} \kappa_1(\nu) I(\nu) d\nu,$$

where f_1 is the relative number of neutral atoms in their first state, r the residual intensity, the other notations being obvious for the reader. For the temperature assumed here we may safely put $f_1 = 1$. Then using the

* $\kappa = \frac{a\sqrt{\pi}}{a\Delta\nu} F\left(\frac{a\Delta\nu}{2}\right)$, where $a = 1{\cdot}6 \times 10^{-2} f.\alpha$, $\alpha = c/v_t\nu_0$ and $F(x)$ is the Gaussian integral. Since $\frac{a\Delta\nu}{2} = 0{\cdot}83$ and $F(0{\cdot}83) = 0{\cdot}76$, we get $\kappa = 1{\cdot}3 \times 10^{-2} cf/v_t\nu_0$.

well-known formula for the atomic absorption coefficient in the case of continuous absorption to the violet of the series limit *

$$\kappa_n(\nu) = \frac{C_n}{\nu^3},$$

where

$$C_n = 64\pi^4 me^{10}/3\sqrt{3}ch^6n^5,$$

and integrating, we get

$$j_{\text{rad.}} = (1-x)r2\pi C_1 \kappa T e^{-\frac{h\nu_1}{kT}} / c^3 m_H.$$

Since for high temperatures $x \sim 1$, we may use the ionization formula for the diluted radiation in the form

$$1 - x = p_e h^3 e^{\frac{h\nu_1}{kT}} / r(2\pi m)^{3/2}(kT)^{5/2}.$$

Thus after simple numerical computations

$$j_{\text{rad.}} = \frac{p_e}{(kT)^{3/2}} \times 2{\cdot}6 \times 10^{-17}. \qquad (1)$$

For lower temperatures this expression should be multiplied by

$$0{\cdot}3T^{5/2}e^{-\frac{h\nu_1}{kT}} \Big/ p_e\left(1 + \frac{0{\cdot}3}{p_e}T^{5/2}e^{-\frac{h\nu_1}{kT}}\right)$$

the factor which takes into account the fact that in the ionization formula x is smaller than unity. The following table gives the values of $j_{\text{rad.}}$ for various T and for $p_e = 10^{-3}$ atm.:—

Radiative Acceleration for Hydrogen Atoms from Ultra-violet Continuous Radiation for $p_e = 10^{-3}$ atm.

T	$j_{\text{rad.}}$
50,000°	$1{\cdot}4 \times 10^3$ C.G.S.
30,000	$3{\cdot}0 \times 10^3$,,
20,000	$5{\cdot}8 \times 10^3$,,
15,000	$8{\cdot}8 \times 10^3$,,
10,000	$4{\cdot}2 \times 10^3$,,
7,000	6 ,,

Further calculation shows that the maximum of $j_{\text{rad.}}$ from ultra-violet continuous radiation is reached for T between 15,000° and 10,000°, *i.e.* approximately for type B5. For higher temperatures $j_{\text{rad.}}$ decreases owing to the decrease in the time spent by the atom in its neutral state, the decrease toward the lower temperatures arising from the rapidly decreasing intensity of ultra-violet radiation. For the temperatures above 10,000°, $j_{\text{rad.}}$ increases as p_e. As a matter of fact, all these computations are valid for the one-sided flux of radiation; the approximation breaks down as soon as the critical value of p_e is reached, where the inward flux of radiation becomes appreciable.

* In a semi-empirical way Menzel showed that this Gaunt's formula is valid for small quantum numbers n (*Lick Bull.*, **17**, 223, 1932).

For some early B stars (except P Cygni stars, which are brighter) we can assume $M_{\text{vis.}} = -4^{\text{m}}\cdot0$, thus for $T = 15{,}000°$, $M_{\text{bol.}} = -5^{\text{m}}\cdot0$. From the mass-luminosity law we infer the mass to be $32\odot$, and then the surface gravity $g = 3.8.10^3$. We observe that in these cases for $p_e = 10^{-3}$ atm. the radiative acceleration from continuous ultra-violet radiation is 2 to 3 times larger than gravitational acceleration; at $p_e = 10^{-2}$ atm. it is 23 times larger.

The acceleration resulting from L_α radiation is

$$j_{\text{rad.}} = \kappa_\alpha \Delta\nu I(\nu)\pi r_\alpha (1 - x)/cm_H,$$

where r_α is the residual intensity in L_α, and $\Delta\nu$ = Doppler width. Using the above given value of k_α and assuming that $r_\alpha = r$ (residual intensity for continuous ultra-violet radiation) we find for $T = 15{,}000°$ and $p_e = 10^{-3}$ atm. that $j_{\text{rad.}}$ is nearly equal to the gravitational acceleration for the stellar atmosphere in question. For L_β the acceleration is considerably smaller.

It seems difficult to escape the conclusion that in some early B atmospheres radiative pressure on hydrogen atoms overbalances gravity.*

The critical layer is at $p_e = 10^{-3}$ atm. and lower, until the layer is reached where the inward flux of radiation becomes appreciable. In order to acquire sufficient initial velocities, the atoms should traverse rather large distances in stellar atmospheres, which requires that the pressure gradient should be small enough. If this gradient is steep, large initial velocities cannot be acquired by the atoms.

The rather large p_e required for the layers, where radiative pressure overbalances gravity for hydrogen atoms, obviously precludes the escape of solitary atoms. There is no difficulty in computing that at such pressures the "molecular horizon" for the escape (in Milne's terminology) should be high above the common horizon, its zenith distance being practically zero. Thus the solitary atoms, accelerated by radiation pressure, should lose their momenta, colliding with the other atoms, which prevent their escape. This is, however, true only for solitary atoms. Having in view the very large partial abundance of hydrogen in stellar atmospheres, we may expect (even without calculating the radiation pressure on the heavier atoms) that these collisions will result only in a redistribution of the acquired momenta amongst the various atoms, the total acquired momentum being practically preserved for the whole layer. Thus eventually the whole layer would start moving. As soon as the dissipation of the higher strata begins, the law of the Bernoullian motion will preserve the flow from below until radiation pressure operates. These considerations, of course, do not pretend to describe the state of affairs in actually dissipating atmospheres; they merely prove that the atmosphere of a B star of assumed luminosity and surface gravity cannot remain in static conditions if the pressure gradient is sufficiently small.

The problem under investigation is an essentially chromospheric problem. Milne and other writers have been dealing with its static variety for calcium and hydrogen atoms. Here, however, we face its new and more general

* Radiative pressure on free electrons due to scattering is considerably larger. The number of escaping electrons is adjusted to the number of escaping ions by the play of electrostatic forces.

dynamic version, relating to the dissipating atmospheres, which can be properly called "non-static chromospheres." The non-static chromospheric problem is obviously much more complicated than the common static problem; its solution, however, opens a wide and new field for investigations, covering such objects as Novæ, WR stars, Be stars and B8–F5 supergiants like α Cygni. The problem to be solved can be formulated as follows: The stellar atmosphere starts its dissipation under radiation pressure. As soon as a considerable amount of matter is ejected it produces an appreciable inward flux of radiation, exerting a sort of braking action on the outflowing atoms. When the optical thickness of the non-static chromosphere increases, this inward flux of radiation can check the further outflow of atoms, thus bringing the non-static chromosphere to an end. In a steady state (if such can be attained) there should be an equilibrium between inertial forces, Newtonian attraction and radiation pressure from both fluxes of radiation. In the following (§ 5) we shall show that for a given stellar model (*i.e.* for a prescribed g and T) a steady non-static chromosphere should satisfy certain necessary conditions. These conditions seem to be fulfilled in P Cygni and some Bse envelopes, showing a permanent outflow of hydrogen atoms. In some cases gravity, being too small for the existence of a static stellar atmosphere, can still be large enough to allow the formation of a permanent non-static chromosphere. The cycle of events can be easily predicted for such stars. The outflow proceeds until the expanding chromosphere is sufficiently thick in dangerous frequencies. As soon as it develops completely, the inward flux of radiation stops the outflow, which begins again as soon as the chromosphere thus set up has dispersed as a detached expanding shell into interstellar space. Then a new cycle of events begins. Such cases correspond to Bne envelopes (§ 8).

The theory of hydrogen non-static chromospheres is given in §§ 4–6. It is shown in § 8 that the characteristics of Bne stars can be explained by postulating the existence of such varying, collapsing and freshly forming non-static chromospheres.

Before proceeding to the solution of the main problem it seems of interest to reconsider critically the other existing hypothesis of Struve regarding the nature of the Be stars.

§ 3. *Some Remarks on the Rotational Hypothesis*

Struve has established that the emission in Be spectra is produced by the nebulosities in some way connected with the central stars—a result which seems to be generally accepted now. But his rotational hypothesis is more disputable.*

Some reasons offered against the rotational hypothesis cannot be maintained, others seem to be rather strong: there are, moreover, some additional weak points in the hypothesis under consideration which should be indicated. It seems therefore of interest to formulate briefly some reasonable objections against the rotational hypothesis.

In the first place it should be mentioned that the empirical correlation

* Compare C. Payne and C. Maulbetsch in *Harv. Circ.*, 364, 1931.

between the widths of emission lines in Bne spectra and the amount of rotation, established by Struve, indicates that the rotation plays an important rôle in the origin of emission in Bne spectra. All objections refer to the special mechanism of the dissipation as suggested by Struve and its applications to the theory of emission contours. The weakest points of the hypothesis in question are as follows :—

(1) The constancy of the rotational momentum requires that linear rotational velocities should decrease as r^{-1}, where r is the distance from the axis. As a characteristic example take H_γ absorption in γ Cass., which is 6·3 A. in width, while emission H_γ in the same star is 4·0 A. (Curtiss). This gives the ratio of r for absorption and emission of 1·6, corresponding to the inverse dilution factor 0·4. On the other hand, the formation of emission lines by recombinations requires that the inverse dilution factor for the lines in question should be much smaller.

(2) If the inverse dilution factor of the order 10^{-3}–10^{-5} is assumed, the rotational velocities in the nebula become too small (a few km. per sec. or even smaller), while the width of the emission lines on Struve's theory requires rotational velocities of the order of 50 km. per sec. or larger.

(3) Rotating rings should produce symmetrical contours, giving no way for explaining asymmetry in emission and its well-known variations.

(4) The rotational hypothesis leaves no hopes for establishing the connection between Bne and P Cygni stars, the emission in these objects being produced by quite different mechanisms.

(5) In some clusters Bse stars are absolutely brighter than Bne stars, though, according to the rotational hypothesis, they should be of the same luminosity.

(6) According to the rotational hypothesis, on the average the total emission in Bne stars should be the same as in Bse stars. It is, however, known that the Balmer emission is much weaker in Bse than in Bne stars.*

(7) The hypothesis under consideration does not explain why in a nebular ring resulting from equatorial instability the rotational velocities for various atoms should be different. It is difficult to see how the effect of distance can enter here if the leading rôle played by radiation-pressure is neglected.

(8) Laplacian rings rotating around the stars can hardly appeal to a modern astrophysicist. Our computations show (§ 2), however, the importance of radiation pressure. Thus even in Laplacian rings radial components of velocities should be at least of the order of their tangential components.

It seems beyond reasonable doubt that the rôle of rotation in Bne stars is very essential. From our point of view the rotation "helps" the birth of a non-static chromosphere—nothing else.

§ 4. *Radiation Field in the Hydrogen Extended Chromosphere*

The Appendix to this paper contains some considerations allowing the use of static formulæ when investigating the field of radiation in a moving chromosphere. It is shown there that the data on Be stars at least do not

* Merrill, *P.A.S.P.*, 45, 49, 1933.

contradict the simplifying supposition on the constancy of the total net flux across a spherical surface. On the other hand, it can be shown that investigating the transfer of radiation through an expanding medium into a steady state one may use the common static equation, provided that the velocity of expansion is moderate.

In dealing with the spherical problem we should use a rather complicated equation of transfer. It seems therefore practical to simplify the problem. For the case of planetary nebulæ this has been done by Milne,* who has reduced the spherical problem to a plane problem, using the boundary condition at the inner surface $i(\theta) = i'(\theta)$. One may ask whether such a condition can be adopted in the case of an expanding medium, extending down to the surface of the star, and which can be properly called an "extended chromosphere." It seems at first that since the heat albedo of the star is unity, the chromospheric radiation falling back on the star would be totally reflected without any loss, thus allowing us to use Milne's boundary condition. This is, however, legitimate only when we are dealing with the integral radiation. When, say, chromospheric L_α radiation penetrates the lower atmosphere it is at least partially "reflected" by the layers, where the collisions and common redistribution in the temperature field play some rôle, so that the "reflected" radiation is to be redistributed amongst other frequencies.

In the spherical problem the following complications arise as compared with the plane problem. First, the constancy of the integral net flux refers to the flux over a spherical surface. Next, the radiation crossing the surface $r = \text{const.}$ at angle θ to the radius should cross the surface $r + dr = \text{const.}$ at angle $\theta + d\theta$. This second complication is of much less importance than the first, since the curvature of the spherical surfaces under investigation is very small. It appears therefore advisable to adopt as a fair compromise some kind of reduction to a plane problem, which takes into account the first property of the spherical problem and neglects the second. This compromise seems to be justifiable in our case, as we do not need to know the distribution of intensity over the nebular surface, but merely the flux in various frequencies at various distances (in order to make the differential equation of motion tractable).

Using the equation $(7a)$ of the "appendix" for the total flow over a spherical surface of the ν-radiation crossing it at angles θ to the corresponding radii, and neglecting the variation in θ, we get

$$\cos\theta \frac{d[i_\nu(\theta) r^2]}{dr} = -\kappa_\nu \rho' i_\nu(\theta) r^2 + E_\nu r^2.$$

Putting $i_\nu(\theta) r^2 = I_\nu(\theta)$ and approaching the solution of the problem by Eddington's method of approximation we obtain

$$\frac{dH_\nu}{d\tau_\nu} = -\bar{I}_\nu + \frac{E_\nu r^2}{\kappa_\nu \rho'}; \qquad \frac{1}{3}\frac{d\bar{I}_\nu}{d\tau_\nu} = -H_\nu, \tag{2}$$

where

$$H_\nu = \tfrac{1}{4}(I_\nu - I_\nu'), \qquad \bar{I}_\nu = \tfrac{1}{2}(I_\nu + I_\nu')$$

* *Zeitschr. f. Aph.*, 1, 98, 1930.

and I_ν, I_ν' are the mean values of $I_\nu(\theta)$ over the hemispheres $0 \leqslant \theta \leqslant \pi/2$ and $\frac{\pi}{2} \leqslant \theta \leqslant \pi$ (κ_ν and ρ' denoting the atomic absorption coefficient and atomic density).

Owing to its relative simplicity and importance for the theory of Be stars, only the problem of a hydrogen "extended chromosphere" will be analysed in the following. Zanstra * has shown that in planetary nebulæ the exciting stellar radiation beyond the Lyman limit (we shall call it "C radiation") is being transformed into the first lines of the Lyman series, which in their turn are being transformed into L_α. In such a manner with the increasing distance from the nuclear star the C-radiation field in nebulæ weakens, being transformed continuously into L_α field, thus producing an intense inward stream of radiation in the corresponding frequencies. Since the absorbing power of hydrogen atoms for these frequencies is very high, L_α radiation will produce dangerous braking action on the outflowing atoms.

In the idealized nebular problem we may neglect L_β, etc., thus supposing that the whole radiation field is composed of C and L_α quanta. Such a simplifying supposition is, however, not legitimate in our case. In the lower part of an expanding chromosphere with the density law $\sim r^{-2}v^{-1}$ the conditions cannot differ much from thermodynamic equilibrium. With increasing r these deviations increase more and more until the nebular conditions are actually reached, where L_β, etc. are being gradually transformed into L_α. Here is the most dangerous region for the stability of the motion, since here the braking effects of L_α are especially powerful. On the other hand, it seems very probable that the emission lines in Be spectra, exhibiting a Balmer decrement of a nebular type, originate in this outer nebular region. It appears therefore practical and reasonable to consider only these nebular strata, where L_β, etc. fields can be neglected in an idealized problem. Of course in doing so one should not forget all the limitations imposed by such an approximation.

It is first necessary to separate the C and the L_α fields and determine the corresponding fluxes. This problem has been treated recently by Ambarzumian,† who solved it for planetaries, using Milne's method of simplifying the spherical problem. Since, as it was pointed out before, the problem of the extended chromosphere is of a different kind, we will solve it in quite a different way. First of all, in the case under consideration it seems artificial to distinguish between "diffuse" and "transmitted" stellar radiation, both kinds of radiation being equally diffuse in the outflowing stream. On the other hand, owing to the failure of the condition $i_\nu(\theta) = i_\nu'(\theta)$ at the inner boundary, the net flux of radiation in our case is not one of the data of the problem, the available data being the incident fluxes of stellar radiation in various frequencies. Thus if the problem of a planetary nebula is analogous to the classical Schwarzschild's problem, the chromospheric case corresponds to Schuster's problem.

Consider first the C-radiation field and define $\bar{\kappa}$, the mean atomic absorp-

* *Ap. J.*, **65**, 50, 1927.

† *M.N.*, **93**, 50, 1932 ; also *Pulkovo Bull.*, **114**, 1933.

tion coefficient for this radiation, by the formula

$$\bar{\kappa} = \int_{\nu_1}^{\infty} \kappa_\nu I(\nu) d\nu \Big/ \int_{\nu_1}^{\infty} I(\nu) d\nu,$$

where $\kappa_\nu = C_1/\nu^3$ and C_1 is known function of atomic constants. It is easy to see that

$$\bar{\kappa} = \frac{C_1}{\nu_1^3}\left[1 + \frac{3kT}{h\nu_1} + 6\left(\frac{kT}{h\nu_1}\right)^2 + \cdots\right]^{-1}$$

Thus, for $T = 20{,}000°$, $\bar{\kappa} = 0{\cdot}68\ C_1/\nu_1^3$.

As a matter of fact the spectral composition of C radiation should depend both on direction and distance, $\bar{\kappa}$ probably decreasing with increasing distance. In order to avoid unnecessary complications we shall neglect all these variations, bearing in mind, however, the necessity of taking them into account in a rigorous theory. Integrating the equations (2) over all frequencies from $\nu = \nu_1$ to $\nu = \infty$, and putting as usual $\tau = \int_{r_0}^{r} \bar{\kappa}\rho' dr$ (ρ' being the atomic density of neutral hydrogen atoms in its first state), we transform these equations to the form

$$\frac{dH}{d\tau} = -\bar{I} + \frac{r^2}{\bar{\kappa}\rho'}\int_{\nu_1}^{\infty} E_\nu d\nu\,; \qquad \frac{1}{3}\frac{d\bar{I}}{d\tau} = -H. \tag{2'}$$

Now $4\pi\int_{\nu_1}^{\infty} E_\nu d\nu = \int_{\nu_1}^{\infty} h\nu n_1(v) dv$, where $n_1(v)$ is the number of captures of v-electrons into the first state per unit volume and time, and $\frac{mv^2}{2} = h\nu - h\nu_1$.

If n_1 is the total number of captures into the first state, we have $4\pi\int_{\nu_1}^{\infty} E_\nu d\nu = n_1 h\bar{\nu}_e$, where $\bar{\nu}_e$ is the mean frequency re-radiated. On the other hand, with our initial notations (i and i') we have for the absorptions

$$2\pi\rho'\int_{\nu_1}^{\infty} \kappa_\nu (i + i') d\nu = 2\pi\rho'\bar{\kappa}\int_{\nu_1}^{\infty} (i + i') d\nu = n^{(1)} h\bar{\nu}_a,$$

$n^{(1)}$ being the number of ionizations from the first state per unit volume and time, $\bar{\nu}_a$ the mean absorbed frequency. Thus

$$\frac{r^2}{\bar{\kappa}\rho'}\int_{\nu_1}^{\infty} E_\nu d\nu = \frac{n_1}{n_1^{(1)}} \cdot \frac{\bar{\nu}_e}{\bar{\nu}_a} \cdot \frac{I + I'}{2}.$$

Since the ionizations from higher states are practically negligible,* in a steady state we may put :

$$n^{(1)} = n_1 + n_2 + n_3 + \cdots$$

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* We infer from the formulæ for κ and $I(\nu)$ (§ 2) that even if the second state is completely metastable (and thus $f_2 : f_1 = 4e^{-\frac{h(\nu_1 - \nu_2)}{kT}}$), $\frac{n_2}{n_1} = \frac{1}{2^3} = \frac{1}{8}$. Actually, owing to the imperfect metastability of the second state, $\frac{n_2}{n_1}$ should be smaller. It goes without saying that the ionization from 3–d, etc. states can be neglected altogether.

On the other hand, since, as we have assumed, the work done by the radiation can be neglected in radiative balance, in a steady state $\bar{\nu}_e = \bar{\nu}_a$.

Denoting further $p = \dfrac{n_1}{n_1 + n_2 + \ldots}$ we get

$$r^2 \int_{\nu_1}^{\infty} E_\nu d\nu / \bar{\kappa}\rho' = p(I + I')/2. \tag{3}$$

The values of $n_1 \ldots$ have been calculated by Cillié *; p is a slowly varying function of T, which for $T = 20{,}000$ is equal to 0·49.

The equations (2′) after using (3) give

$$\frac{d(I - I')}{d\tau} = -2(I + I')(1 - p)$$

$$\frac{d(I + I')}{d\tau} = -\tfrac{3}{2}(I - I'), \tag{4}$$

which after differentiation and obvious substitution gives

$$\frac{d^2 I - I'}{d\tau^2} - 3(1 - p)(I - I') = 0,$$

with the general solution

$$I - I' = Ae^{-\lambda\tau} + Be^{\lambda\tau}$$

where $\lambda = \sqrt{3(1 - p)}$. Now from the first equation (4)

$$I + I' = \frac{3}{2\lambda}(Ae^{-\lambda\tau} - Be^{\lambda\tau}),$$

whence

$$2I = Ae^{-\lambda\tau}\left(1 + \frac{3}{2\lambda}\right) + Be^{\lambda\tau}\left(1 - \frac{3}{2\lambda}\right)$$

$$2I' = -Ae^{-\lambda\tau}\left(1 - \frac{3}{2\lambda}\right) - Be^{\lambda\tau}\left(1 + \frac{3}{2\lambda}\right).$$

The choice of the inner boundary is somewhat arbitrary, the only condition being that it should be located in some layer of the stellar chromosphere where the collisions and thermal redistributions of frequencies are negligible. Without specifying it in detail we impose the following boundary conditions

$$I = S \text{ at } \tau = 0 \quad \text{and} \quad I' = 0 \text{ at } \tau = \tau_1,$$

τ_1 being the total optical thickness of the upper chromosphere for C radiation (more accurately for effective ν). These conditions give

$$A = 2S\left(1 + \frac{3}{2\lambda}\right)e^{\lambda\tau_1} \Big/ \left(1 + \frac{3}{2\lambda}\right)^2 e^{\lambda\tau_1} - \left(1 - \frac{3}{2\lambda}\right)^2 e^{-\lambda\tau_1};$$

$$B = -2S\left(1 - \frac{3}{2\lambda}\right)e^{-\lambda\tau_1} \Big/ \left(1 + \frac{3}{2\lambda}\right)^2 e^{\lambda\tau_1} - \left(1 - \frac{3}{2\lambda}\right)^2 e^{-\lambda\tau_1},$$

* *M.N.*, **92**, 820, 1932.

$I - I'$ decreasing continuously as τ increases, without passing through a minimum.

Consider now the L_α radiation. Let I_α, I_α' have for this radiation the same meaning as I and I' for the C frequencies. The condition of radiative equilibrium for total radiation, or rather the constancy of the total net flux C across any spherical surface gives *

$$I_\alpha - I_\alpha' = C - (I - I'). \tag{5}$$

Thus the net flux in L_α changes its direction for τ, satisfying the condition $I - I' = C$. As before, we have for L_α radiation

$$\frac{d(I_\alpha + I_\alpha')}{d\tau_\alpha} = -\tfrac{3}{2}(I_\alpha - I_\alpha'), \tag{6}$$

where according to the definition $\tau_\alpha = \tau \cdot \bar{\kappa}_\alpha/\bar{\kappa} = \frac{\tau}{\omega}$, ω being very small. In fact, κ_α averaged over the Doppler width for $T = 20{,}000°$ is $3{\cdot}6 \times 10^{-14}$, whilst $\bar{\kappa} = 5{\cdot}4 \times 10^{-18}$; whence $\frac{1}{\omega} = 6{\cdot}7 \times 10^3$.

After integration we get from (5) and (6)

$$I_\alpha + I_\alpha' = -\tfrac{3}{2}C\frac{\tau}{\omega} - \frac{3}{2}\frac{A}{\omega\lambda}e^{-\lambda\tau} + \frac{3}{2}\frac{B}{\omega\lambda}e^{\lambda\tau} + D,$$

$$I_\alpha - I_\alpha' = C - Ae^{-\lambda\tau} - Be^{\lambda\tau},$$

where D is a new constant of integration. Whence

$$2I_\alpha = C\left(1 - \frac{3}{2\omega}\tau\right) - A\left(1 + \frac{3}{2\lambda\omega}\right)e^{-\lambda\tau} - B\left(1 - \frac{3}{2\lambda\omega}\right)e^{\lambda\tau} + D,$$

$$2I_\alpha' = -C\left(1 + \frac{3}{2\omega}\tau\right) + A\left(1 - \frac{3}{2\omega\lambda}\right)e^{-\lambda\tau} + B\left(1 + \frac{3}{2\lambda\omega}\right)e^{\lambda\tau} + D.$$

The boundary conditions for L_α radiation are

$$I_\alpha = S_1 \text{ at } \tau = 0 \quad \text{and} \quad I_\alpha' = 0 \text{ at } \tau = \tau_1;$$

thus finally with sufficient accuracy

$$C = \frac{1}{\tau_1\lambda}[A(1 - e^{-\lambda\tau_1}) + B(e^{\lambda\tau_1} - 1)] + \tfrac{4}{3}S_1\frac{\omega}{\tau_1}$$

neglecting in parentheses all the terms not containing $\frac{1}{\omega}$ and putting $\frac{4}{3} + \frac{\tau_1}{\omega} = \frac{\tau_1}{\omega}$, which is always permissible if τ_1 is not extremely small (of the order of 10^{-4}). Owing to the factor ω, the term with S_1 is relatively small and can be neglected. It is seen therefore that chromospheric radiation

* Some energy is lost from the field under consideration by transformation of C radiation into lower frequencies other than L_α. The upper limit of this relative loss corresponding to the complete transformation of C field is about 30 per cent. of the field energy.

field is determined practically by stellar C radiation, stellar L_α being of no importance. On the other hand, owing to the large factor $\frac{1}{\omega}$, chromospheric L_α radiation is considerably more intense than corresponding C radiation. The net flux in L_α, i.e. $I_\alpha - I_\alpha'$, does not contain the factor $\frac{1}{\omega}$, and is much smaller than I_α and I_α' taken separately.

So far we have been concerned only with integral L_α radiation. As a matter of fact, inside the chromosphere this radiation is not monochromatic, being confined, owing to the thermal velocities, to a certain finite interval in λ, the structure of which is of a highest mechanical importance. At a given τ the atoms are moving with systematic velocity $v(\tau)$. If excited by L_α radiation emitted by the atoms moving with a different velocity, they should absorb in a special way, which can be determined only after performing the Lorentz transformation from one reference frame to another. This transformation would make the problem extremely complicated, as the effective L_α line would be composed of the contributions of the atoms having various systematic velocities v, somewhat smeared out due to the thermal spread. In principle this problem admits a solution, but it leads to the equation of motion of a very complicated integro-differential type. Fortunately, however, the very conditions of the problem allow us to avoid these difficulties.

Consider an extreme case : a systematic velocity decreasing from 250 km. ($\tau = 0$) to zero ($\tau = \tau_1$) with $\tau_1 = 10$—which seems to be a rather conservative estimate. On the average a decrease in the velocity of amount $\Delta v = 0{\cdot}1$ km. corresponds to an increase of $\Delta\tau = 4{\cdot}10^{-3}$ or (considering the optical depth for L_α) $\Delta\tau_\alpha = 27$. Thus, independently of thermal spread in velocities, L_α is practically opaque when dealing with the distances corresponding to small variations in systematic velocity. This makes the atoms depending on L_α emitted by the neighbouring atoms have the same systematic velocity. The transformation from one reference frame to another becomes unnecessary, which enormously simplifies the mechanical problem. It seems of interest to note that for a detached expanding envelope the boundary condition $I_\alpha = S_1$ at $\tau = 0$ may not refer to the stellar L_α. If the relative velocity "inner boundary – star" is large enough, the atoms at $\tau = 0$ will absorb outside the line in the continuous spectrum of the star.

We observe in passing that in contradistinction to the solution of the nebular problem the total net flux C (considered as constant) decreases to zero for large τ_1, all radiation being returned to the star and re-radiated in a steady state all along the spectrum. This should be expected since the problem under investigation is essentially a Schuster's problem.

Considering all these results one should not forget the simplifying conditions imposed at the very beginning. In fact, instead of the L_α field, one should consider the combined Lyman fields. It is impossible to reach any conclusions by simply putting in the foregoing formulæ, *e.g.*, the combined $L_\alpha + L_\beta$ intensities instead of L_α intensity, since in this case the ratio of absorption coefficients ω should vary with τ, depending upon the relative

densities of L_α and L_β fields. The absorption coefficient for L_β is 6 times smaller than k_α. Thus, using our results for investigating the mechanical action of radiation, we actually get its upper limit. Owing to the presence of L_β radiation the integral effect should be smaller by a factor between 1 and 1/6 (extreme values); the mean value of this factor being 0·3 say. In such a way the results of the next paragraph should be qualitatively valid even if L_β radiation cannot be neglected.

§ 5. *On the Mechanical State of a Non-static Chromosphere*

The differential equation of motion for the hydrogen atoms moving in the Newtonian field of the star and the resulting chromospheric C and L_α fields of radiation can be written for our simplified problem as follows:— *

$$\frac{dv}{dt} = -\frac{GM}{r^2} + \frac{\pi(1-x)}{m_H c}\int_{\nu_1}^{\infty} \kappa(\nu)[i(\nu) - i'(\nu)]d\nu + \frac{\pi(1-x)}{m_H c}\int_{-\infty}^{+\infty} \kappa_\alpha(\nu)[i_\alpha(\nu) - i_\alpha'(\nu)]d\nu. \quad (7)$$

We have neglected the gravitational field of the chromosphere and the hydrostatic pressure gradient and have assumed that at the temperatures in question practically all the H atoms are in their first state. M is the mass of the star, $(1-x)$ the relative number of neutral atoms and m_H the mass of the H atom. Introducing mean values and recalling that $i = I/r^2$, we get

$$r^2\frac{dv}{dt} = -GM + \frac{\pi(1-x)}{m_H c}\bar{\kappa}(I - I') + \frac{\pi(1-x)}{m_H c}\bar{\kappa}_\alpha(I_\alpha - I_\alpha'). \quad (7')$$

It is convenient to transform this equation to a new variable τ. The left-hand term is obviously †

$$r^2 v\frac{dv}{dr} = r^2 v\frac{dv}{d\tau}\bar{\kappa}N(1-x) = \mu(1-x)\bar{\kappa}\frac{dv}{d\tau}, \quad (8)$$

where N is the total number of hydrogen atoms per unit volume and $4\pi\mu = 4\pi r^2 N v$ is the total number of hydrogen atoms ejected from the star per second (by the equation of continuity). Thus by (7′) and (8)

$$\mu\frac{dv}{d\tau} = -\frac{GM}{\bar{\kappa}(1-x)} + \frac{\pi}{m_H c}\left[I - I' + \frac{1}{\omega}(C - I + I')\right] = -\frac{GM}{\bar{\kappa}(1-x)} + \frac{\pi}{m_H c\omega}(C - I + I') \quad (9)$$

with a fair degree of approximation.

* Let K_0 be the reference frame at rest with respect to the star, and K another reference frame at rest with respect to moving atoms. The total momentum gained by the atoms is composed of two contributions; one produced by absorptions and the other communicated by emissions. In the K frame the last momentum is zero; however, owing to the asymmetry in emission, it will not be zero for the K_0 frame (*cf.* Milne, *M.N.*, **86**, 584, 1926). Now the Galilean principle is valid with respect to the total momentum communicated to the swarm; in the K_0 frame it will be therefore equal to the momentum communicated by the absorptions in the K frame.

† Actually $\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}$, and here $\frac{\partial v}{\partial t} = 0$ since the state is assumed steady. Derivatives with respect to r or τ can then conveniently be written as total ones.

Now

$$2\pi(1-x)\int_{\nu_1}^{\infty}\psi(\nu)(i+i')d\nu = Nx^2\Sigma n_i,$$

$$2\pi(1-x_0)\int_{\nu_1}^{\infty}\psi(\nu)(i_0+i_0')d\nu = N_0x_0^2\Sigma n_i,$$

where $\psi(\nu)$ is the probability function for ionization and n_i the total number of captures into i-th state. Since, as is assumed, the spectral composition of C radiation does not vary with the distance, the ratio of the mean absorption coefficient to mean $\psi(\nu)$ should be approximately constant throughout the chromosphere. Hence

$$(1-x)(I+I')/(1-x_0)(I_0+I_0') = Nx^2r^2/N_0x_0^2R_0^2.$$

Making use of the equation of continuity we get finally

$$x^2/1-x = \frac{x_0^2}{1-x_0}\frac{I+I'}{I_0+I_0'}\frac{v}{v_0}, \tag{10}$$

where x_0, R_0, v_0 refer to the inner boundary of the nebular layer. The formula (10) is nothing else than the ionization formula for a non-static chromosphere. In what follows, having in view the high temperature of the exciting star, we shall put $x^2 = x_0^2 = 1$. Of course this does not mean that the ionization in the chromosphere is assumed to be constant.

For the applications of the theory it seems necessary to select a certain stellar model defined by its effective temperature T, surface gravity and radius. In the cases under consideration, when the density falls down so slowly, it seems difficult to specify both photospheric and reversing layers. However, even in these cases there should be some strata analogous to the photosphere and reversing layer of the static star. Elementary derivation of the "stellar radius" in terms of T and luminosity permits us to find the value of R_*, which for the stars under consideration would not necessarily coincide with the radius of the reversing layer. Being perfectly aware of this difficulty we shall, however, when necessary in what follows use R_* also as the radius of the reversing layer, the error committed being of no importance for such a rough calculation.

Putting

$$u = \mu v/R_*^2, \qquad u_0 = \mu v_0/R_*^2, \qquad A_1 = A/R_*^2 \quad \text{and} \quad B_1 = B/R_*^2$$

we get the final equation of motion

$$\frac{du}{d\tau} = -g_*\frac{(A_1e^{-\lambda\tau} - B_1e^{\lambda\tau})}{\bar{\kappa}(1-x_0)(A_1-B_1)}\frac{u}{u_0} + \frac{\pi}{m_Hc\omega}\left[\frac{1}{\tau_1\lambda}[A_1(1-e^{-\lambda\tau_1}) + B_1(e^{\lambda\tau_1}-1)] - (A_1e^{-\lambda\tau} + B_1e^{\lambda\tau})\right], \tag{11}$$

where g_* is the gravity at the stellar surface. This equation gives u as a function of τ for assumed τ_1. In order that a non-static chromosphere with the prescribed τ_1 and μv_0 be possible it is necessary and sufficient that the

integral of the final equation be positive in the τ-domain from $\tau = 0$ to $\tau = \tau_1$, zero value being allowed only for $\tau = \tau_1$.

The general solution of (11) is

$$u = u_0 e^{-\int_0^\tau f(\tau)d\tau} + e^{-\int^\tau f(\tau)d\tau} \int_0^\tau \psi(\tau) e^{\int f(\tau)d\tau} d\tau, \tag{12}$$

where

$$f(\tau) = g_*(A_1 e^{-\lambda\tau} - B_1 e^{\lambda\tau})/\bar{\kappa}(1 - x_0)(A_1 - B_1)u_0,$$

$$\psi(\tau) = -\frac{\pi}{m_H c\omega}\left[\frac{1}{\tau_1\lambda}(A_1(1 - e^{-\lambda\tau_1}) + B_1(e^{\lambda\tau_1} - 1)) - (A_1 e^{-\lambda\tau} + B_1 e^{\lambda\tau})\right],$$

the constants A_1 and B_1 being known functions of S and τ_1.

For a prescribed τ_1 the solution (12) gives the values of $u_0 = \mu v_0/R_*^2$ for which the non-static solutions of the chromospheric problem are possible. But this does not imply that all values of μ and v_0 satisfying the condition $\mu v_0 = u_0 R_*^2$ satisfy the problem. It is easy to see that there exists a supplementary equation connecting τ_1 and μv_0.

In fact, we have by (10)

$$d\tau = \bar{\kappa}N(1 - x)dr = \bar{\kappa}\frac{Nv_0(I_0 + I_0')}{v(I + I')}(1 - x_0)dr,$$

or making use of the equation of continuity

$$v^2(I + I')d\tau = \bar{\kappa}(I_0 + I_0')(1 - x_0)v_0\mu dr/r^2.$$

Multiplying by $\frac{\mu^2}{R_*^4}$ and integrating from $\tau = 0$ to $\tau = \tau_1$ and from $r = R_0$ to $r = \infty$, we get *

$$\int_0^{\tau_1} u^2(I + I')d\tau = \bar{\kappa}(I_0 + I_0')(1 - x_0)v_0\mu^3/R_*^4 R_0. \tag{13}$$

Thus for a given S, g_*, R_* and R_0 a non-static chromospheric configuration depends on two parameters τ_1 and μ, or on μ and v_0. In the μv_0 plane possible steady chromospheric configurations are confined to a plane area Σ limited by a certain curve $\mu = \chi(v_0)$, every point in this domain representing a non-static configuration. If on account of some unknown physical changes in the star μ or v_0 changes, the representative point displaces in the Σ domain. As soon as it crosses the limiting curve, the steady non-static configuration ceases to be possible.

Considering special cases, it seems convenient to use a new parameter $a = g_*/\bar{\kappa}(1 - x_0)u_0$, which for a given star is $\sim (\mu v_0)^{-1}$.

(1) Small τ_1. Here ψ is always positive, and steady configurations are possible also for large a, *i.e.* for relatively small μv_0.

(2) Large and moderately large τ_1. Here we can put $B_1 e^{\tau\lambda} \sim 0$ for all τ. The solution degenerates into

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* If integrated to $\tau = \tau$ and $r = r$ this equation would give a finite relation between r and τ, *i.e.* the integral of the equation connecting dr and $d\tau$, the other independent differential equation of the problem being (7').

$$u = u_0 e^{-\frac{a}{\lambda}(1-t)} + \frac{b}{\lambda}\left[-\frac{\lambda}{a} + \frac{\lambda}{a} e^{-\frac{a}{\lambda}(1-t)} + \frac{1}{\tau_1 \lambda} E\left(-\frac{a}{\lambda}\right) e^{\frac{a}{\lambda}t} - \frac{1}{\tau_1 \lambda} E\left(-\frac{a}{\lambda}t\right) e^{\frac{a}{\lambda}t} \right], \quad (14)$$

where E is the exponential integral, $t = e^{-\lambda\tau}$ and $b = 2\pi S/R_*^2\left(1 + \frac{3}{2\lambda}\right) m_H c\omega$, S being proportional to R_0^2. Were the space between the "photosphere" and the R_0-layer transparent, S/R_*^2 would be simply the intensity of the photospheric C radiation per unit surface ($b = 3 \cdot 10^{28}$ for $T = 20{,}000°$). Actually S/R_*^2 should be smaller by the amount of absorption in the underlying strata.

If a and $ae^{-\lambda\tau_1}$ are large we get after expanding E in series $u = \frac{b}{a}\left[\frac{e^{\tau\lambda}}{\tau_1\lambda} - 1\right]$ for $\tau >> \frac{1}{a}$. Since u is negative for $\log \tau_1\lambda >> \tau_1\lambda >> \frac{\lambda}{a}$, steady non-static chromospheres for large g_* or small μv_0 are impossible.

If a is very small we get from (14)

$$u = u_0 + \frac{b}{\lambda}\left[e^{-\tau\lambda} - 1 + \tau/\tau_1 \right]. \quad (14')$$

In this case a steady configuration is possible if $u_0 > \frac{b}{\lambda}\left[1 - \frac{1}{\tau_1\lambda} - \frac{\log \tau_1\lambda}{\tau_1\lambda}\right]$ which is the maximum value of brackets in (14'). If this condition is satisfied $\frac{du}{d\tau} < 0$ for small τ_1, changes its sign for $\tau = \log \lambda\tau_1/\lambda$, and then remains positive. This change in the direction of acceleration is due to the change in the direction of net flux in L_α.

(3) Very large τ_1. Here also ψ is negative for all τ. Thus for very large τ_1 a steady non-static chromosphere cannot exist for large a, *i.e.* for relatively small μv_0. On the other hand, if a is not very small, the equation (14) in this limiting case takes the form

$$u = u_0 e^{-a(1-t)/\lambda} + \frac{b}{a}\left[e^{-a(1-t)/\lambda} - 1 + \frac{a}{\lambda} f(t) \right], \quad (14'')$$

where $f(t) = -E\left(-\frac{a}{\lambda}t\right)\lambda t$ is very small for moderate τ and tends to unity for $\tau \to \tau_1$. If a is very small we come back to (14').

These considerations show the theoretical possibility of the following three types of steady configurations :—

(I) Velocity decreases monotonically from $\tau = 0$ to $\tau = \tau_1$.
(II) Velocity decreases, and then after passing through a minimum begins to increase.*
(III) Configurations corresponding to small τ_1.

* Equation (14) corresponds to cases (I) and (II) only. It is easy to see that here $\frac{du}{d\tau}$ is negative for all τ, or it is at first negative and then positive.

In astrophysical applications case III corresponds to weak emission lines, which become perceptible only if the spread in velocities is sufficiently small. We can connect this configuration with some Be stars showing weak and narrow lines, centred on underlying absorption.

The most interesting are configurations corresponding to large τ_1, for it seems almost certain that in the stars of P Cygni and Bne types the hydrogen chromospheres are almost opaque beyond the Lyman limit. Such chromospheres, if they are in a steady state, can belong to the configurations of I and II types. The type of configuration depends here on the value of a. If a is very small, the acceleration can change its sign, which is due to the change in the direction of the net flux in L_α. This flux represents a certain "barrier," which the atoms should "climb" in order to escape the star altogether. In general the τ corresponding to the "crest" of this barrier is close (but not equal) to the root of the equation $I_\alpha - I_\alpha' = 0$, or $I - I' = C$. If a is not very small the descending slope of the "barrier" is insufficient to change the sign of acceleration and the velocity proceeds to decrease.

There will be, however, in operation another cause which has a tendency to counterbalance somewhat the effects of acceleration as calculated above. This is the force analogous to that considered by Milne in his theory of high chromospheric and Novæ ejection. Consider the atoms moving in the accelerating field of chromospheric L_α radiation, C radiation being neglected. Those of them which acquire random velocities in the direction of the flow will move into the violet wing of the chromospheric L_α, where the net accelerating flux is less intense ; their relative velocities will decrease, and they will be finally brought back to the centre of the line. On the other hand, the atoms which acquire the random velocities in the opposite direction should be further decelerated. It is easily understood that if the L_α field is decelerating, quite an opposite effect is to be expected. If the mean time between consecutive atomic collisions is larger than (or of the order of) that corresponding to the consecutive absorptions of $L\alpha$ quanta, the effect in question is of no importance. On the other hand, in the outermost parts of the nebula, where the collisions are quite negligible, this "contour effect" can produce in a statistical way an additional apparent force acting against the general field gradient.

§ 6. *P Cygni Stars*

These stars exhibit a permanent outflow of matter, absorption lines originating in the "reversing layer" being displaced toward the violet, while the centres of emission lines seem to be undisplaced. The half-width of the emission band corresponds to the maximum velocity attained in the nebular strata. Comparing it with the velocity corresponding to the absorption line, we infer (even taking into account the blending effect) that the atoms after the ejection are decelerated before reaching distances where the dilution of radiation allows the formation of the observed emission lines.

In order to apply the foregoing theory a certain model of P Cygni should be adopted (*i.e.* G_*, T and R_*). This star is abnormal as showing a spectro-

photometric temperature much lower than the one required by the observed ionization and excitation. Since we are dealing here with the frequencies in the far ultra-violet it seems safe to adopt some value of T between 25,000° and 15,000°—we choose 20,000°, which is close to the value determined by Zanstra's method.*

According to Shapley, the mean photographic absolute magnitude of P Cygni stars in the Large Magellanic Cloud is $-5^{m}\cdot4$ (with revised zero point). Allowing for the selective character of the data we may suggest a somewhat lower luminosity for these stars, adopting provisionally an absolute visual magnitude of $-5^{m}\cdot0$. Since in visual light P Cygni shows a spectrophotometric gradient corresponding to 6000°, we can determine R_* directly by comparing the star with the Sun. This gives $R_* = 100R\odot$. Slightly extrapolating the mass-luminosity relation we adopt the mass $50\odot$ and then $g = 1\cdot3\cdot10^2$.

As was said before, the optical thickness, τ_1, should be large. In this case (14′) and (14″) suggest that the maximum of u corresponds to the lower boundary of the nebular region. Now since the half-width of the H-emission in P Cygni is 150 km., we may adopt $v_0 = 150$ km. In the preceding section we have computed $b = 3\cdot10^{28}$ (for $T = 20{,}000°$) without taking into account the absorption in the underlying layers. Allowing for this absorption we take the residual intensity $b = 3\cdot10^{27}$ (a factor 0·1). This reduction in the intensity of the stimulating radiation would slightly increase the temperature derived by Zanstra's method without changing it materially.

According to a well-known theorem the conditions of ionization in nebular matter (density ρ) are the same as in the material of density $\rho\delta$ (δ is the factor of dilution) in thermodynamic equilibrium. By the equation of continuity we have $\rho_0 = \rho_* R_*^2 v_* / R_0^2 v_0$, the suffixes $_*$ and $_0$ referring to the "reversing layer" and the lower boundary of the nebular region. Now at $r = R_0$ the factor of dilution $\delta \geqslant R_0^2/R_*^2$ and thus $\rho_0\delta \geqslant \rho_* v_*/v_0 > \rho_*$. It follows that for the isothermal case the ionization at R_* is larger than at R_0. Adopting $p_{e*} = 10^{-6}$ atm.† and for the assumed temperature $1 - x_* = 1\cdot4 \times 10^{-7}$ we get $a < g_*/\bar{\kappa}(1 - x_*)u_0$ or $au_0 < 1\cdot4 \times 10^{26}$.

Passing to (14″) and imposing the condition that $u \geqslant 0$ for $\tau = \tau_1$ we easily see that $au_0 > b$, while according to the above inequality $au_0 < 1\cdot4 \times 10^{26}$. Both conditions can be reconciled only if the outward stream of C radiation over a spherical surface is reduced by the factor $5\cdot10^{-3}$. Such a large reduction would, however, contradict the rather successful application of Zanstra's method to this star. It appears that only the equation (14′) for large τ_1 and small a (chromosphere of the II type) does not contradict the numerical conditions of the problem. Since the adopted value of τ_1 is of no practical importance, we take $\lambda\tau_1 = 10$, giving $\lambda\tau' = 2\cdot3$ for the minimum u reached in the chromosphere. It seems reasonable to suppose that $u_{\min}$ is of the order of u_0 itself, say $0\cdot1u_0$ (again the adopted value of $u_{\min}$ is of no importance here).

* Beals, *M.N.*, **92**, 677, 1932.

† Using Unsöld's method, Miss Payne finds for static Bo stars $p_e = 10^{-5}$ to 10^{-6}. (*The Stars of High Luminosity*, p. 94.)

After these preliminaries we easily find $u_0 = 1{\cdot}8{\cdot}10^{27}$, $a < 8{\cdot}10^{-2}$ and (for adopted R_*) $\mu = 6{\cdot}10^{45}$ H-atoms per second. As a matter of fact, this is only the upper limit. Taking into account the possible $L\beta$ field it seems reasonable to correct this value by the factor 0·3 or so, which gives $\mu = 2{\cdot}10^{45}$ H-atoms per second or $4\pi\mu = 1{\cdot}2 \times 10^{30}$ gr. per year, *i.e.* 10^{-5} M_* if $M_* = 50\odot$.

For a very rough check of this result we may use the formula $4\pi\mu = 4\pi\rho_* v_* R_*^2$. The displacement of the absorption line—v_*—is of the order of 220 km. For $p_{e*} = 10^{-6}$ atm. and the adopted temperature $\rho_* = 6{\cdot}10^{-13}$; this gives $4\pi\mu = 8 \times 10^{21}$ gr. per second or $\mu = 4{\cdot}10^{44}$ H-atoms, which may be favourably compared with the value resulting from the above theory.*

To be certain that the proposed solution satisfies all the conditions of the problem, we represent the formula (13) as follows :—

$$\int_0^\tau u^2(I + I')d\tau / I_0 + I_0' > \bar{\kappa}(1 - x_*)R_*^2 u_0^3 / v_0^2 R_0.$$

Since

$$u_0 = \mu v_0 / R_*^2 \quad \text{and} \quad 1 - x_* < 1 - x_0,$$

direct integration gives for the left-hand term the value $0{\cdot}17{\cdot}10^{54}$. Using the adopted values of the other parameters we find $R_0 > 1{\cdot}9{\cdot}10^{15}$, or $R_0/R_* > 3{\cdot}10^2$. In this way the inverse factor of dilution comes out to be $\delta^{-1} < 10^{-5}$, a not unreasonable amount of dilution for the production of emission lines. The density at $r = R_0$ is less than $9{\cdot}10^{-17}$ c.g.s. units.

We may conclude that the above theory gives a rather satisfactory representation of the P Cygni phenomena. The velocity of outflow at $p_{e*} = 10^{-6}$ is 220 km./sec., then it decreases continuously and at the lower boundary of the nebular region it is 150 km./sec. It continues to decrease until τ, corresponding to the minimum velocity, is reached; then it begins to increase. The velocity at infinity ($\tau = \tau_1$) is only slightly smaller than $v_0 = 150$ km./sec. Supposing that the ejection started in 1715 after the last slow increase of brightness, we may conclude that P Cygni has lost $2{\cdot}10^{-3}$ of its mass. It goes without saying that such a large loss of mass cannot continue for a very long time.

§ 7. *On the Acquisition of Initial Velocities. The Forerunners*

The above analysis shows that if certain initial conditions are fulfilled for a given g_* and T a non-static chromosphere can exist as a steady though not necessarily stable formation. This analysis is correct as far as nebular layers of the chromosphere (where the bright lines originate) are concerned. As a matter of fact, a non-static chromosphere taken as a whole should be much more complicated. It starts at the photospheric layers, where the collisions predominate, its density falling down very slowly according to the law

* The velocity in the chromospheres of the II type cannot be represented as $v \sim r^n (n < 0)$. It is therefore difficult to say whether in this case the total net flux in C and La frequencies can be considered as constant (see Appendix). A second approximation is evidently needed, which, however, hardly would change materially the order of the resulting quantities.

$\rho \sim v^{-1}r^{-2}$. Higher up in the lower chromospheric layers ordinary scattering processes take place without interference of collisions. In order to study the state of motion of these layers it is necessary to take into account all radiation fields capable of being produced by hydrogen atoms. As the distance from the star increases, the conditions become more and more similar to those in nebulæ, where Zanstra's hypothesis on the prevalence of the L_α field is valid. The results of § 5 (and especially the analysis of the "limiting curves") give the necessary and sufficient conditions for the existence of steady flow in nebular layers, but they do not indicate how the necessary initial velocity—v_0—can be acquired in the lower layers. Thus for a non-static chromosphere taken as a whole the above conditions are necessary, but are not sufficient. They only show that if such a steady dynamic formation exists, it should belong to one of the types discussed above. It is outside the scope of this paper to analyse how this initial velocity—v_0—is really acquired, which is a very perplexing question.

No large initial velocity can be acquired by the atoms subject to the intense inward flux of L_α and L_β, etc. radiations. "The gun and powder" should be looked for in the lower layers, where on account of the radiative redistribution and collisions L_α, L_β, etc. radiations are to be transformed into the frequencies less dangerous for the absorbing atoms. This expectation is in conformity with the results of § 2, which show that the radiation pressure produced by a one-sided flux overbalances the force of gravity by several times only in the layers where the electron pressure is high enough (10^{-3} atm.).

It is seen from the equation of continuity that the pressure gradient in the non-static case is very small, thus leading to very high resulting velocities. Such velocities can, however, be produced only by a one-sided flux; if the flow is steady, the braking action of the inward flux in the various frequencies puts a rather severe limit to the acquired velocities. Even more, this inward flux can stop the outflow altogether, thus bringing the non-static chromosphere to an end.

It seems of importance for the following to study the initial motion in greater detail. We consider the atoms starting from the stellar surface as forerunners of the freshly formed moving strata of the non-static chromosphere. These atoms are subject to a one-sided flux of radiation, and their motion can easily be determined. Taking into account C radiation only and putting in the ionization formula $x = 1$ as before, we infer from § 2 that the radiative acceleration—$j_{\text{rad.}}$—is independent of the dilution factor. Using (1) and the equation of continuity, we find

$$j_{\text{rad.}} = Cp_{e*}v_*R_*^2/(kT)^{\frac{3}{2}}vr^2 = C'/vr^2,$$

where p_{e*}, v_* and R_* refer to the "stellar surface," C' is a new constant. The total acceleration for radial flow is

$$j = C'/vr^2 - C''/r^2,$$

where $C'' = GM$. The equation of motion is obviously

$$v\frac{dv}{dr} = C'/vr^2 - C''/r^2,$$

which gives after integration

$$\frac{1}{r} - \frac{1}{R} = \frac{1}{C''}\left[\frac{(v-a)^2}{2} - \frac{(v_*-a)^2}{2} + 2a(v-a) - 2a(v_*-a) + a^2 \log\frac{v-a}{v_*-a}\right],$$

with a new constant

$$a = C'/C'' = \frac{Cp_{e*}v_*{R_*}^2}{(kT)^{\frac{3}{2}}GM}.$$

From the above expression we infer that if $v_* > a$, $v > a$ for all r and t, and *vice versa*. The velocity at infinity differs from a very slightly : practically the velocity a is reached pretty soon.

For the B star model of § 2 and $p_{e*} = 10^{-3}$ atm. we have $C = 2{\cdot}6 \times 10^{-17}$ and $a = 2v_*$. For rotating stars the effective g_* should be smaller and the limiting v larger for the same p_{e*} and v_*. We observe these forerunners in Novæ soon after outbursts. In these cases probably $a < v_*$ (v_* corresponding to the velocity of the displaced absorption line, steadily decreasing in the course of a few weeks).

§ 8. *Bne Stars*

A straightforward application of the foregoing theory to Bne stars does not look very promising. There are two difficulties here ; one of them seems to be unsurmountable.

(1) It is known that Be stars taken as a whole are more luminous than absorption B stars. The existing data, crude as they are, suggest for type B3 the difference of $1^m.5$.* Thus we can take $-3^m{\cdot}0$ as an upper limit to the absolute magnitude of B3e stars. On the other hand, the galactic clusters data show that Bne stars are less luminous than Bse. We shall not be therefore too far from the truth by taking $-2^m{\cdot}5$ as the upper limit to the mean absolute magnitude of B3ne stars. For $T = 15{,}000°$ this gives a value of gravity which is larger than critical $j_{\text{rad.}}$ for $p_e = 10^{-3}$ atm. (see § 2). Thus the radiative acceleration here can hardly even balance gravity. The matter looks even worse for slightly higher temperatures.

We cannot help thinking that in Bne stars radiative pressure, if working alone against gravity, cannot produce such a dissipation of the atmospheres as is actually suggested by the observations. Here Struve's hypothesis of the rapid rotation as manifested by the width of the absorption lines is of value. Rapid rotation working against the gravity evidently can sponsor the dissipation, even if the luminosity is not exceedingly high. In these cases the lines of flow are spirals, and the chromosphere is limited externally by two co-axial cones, their axis being that of stellar rotation.

(2) No well-expressed asymmetry—"absorption wings-broad emission"—has been observed in the spectra of Bne stars, thus indicating that the initial velocity in the reversing layer is small, smaller, say, than 20 km. On the

* Revised value by Gerasimovič, in press.

other hand, the width of emission lines suggests that in the nebular regions rather high velocities are attained (sometimes as big as 100–150 km.). It seems, therefore, that after leaving the stellar surface the atoms are subject to radial acceleration. The considerations concerning I and II type chromospheres show, however, that this acceleration should be created between R_* and R_0 layers. On the other hand, the intensity of emission rules out the possibility of small τ_1 in these envelopes, thus making unlikely that these velocities are acquired in the lower strata. It seems, therefore, improbable that Bne envelopes represent non-static chromospheres of a steady type.

This reasoning is fully supported by the behaviour of the Bne emission. In contrast to P Cygni and some Bse, the stars under consideration exhibit cyclically varying emission, this phenomenon being quite a general characteristic of the whole family. We may think that such a non-steady outflow proceeds until the expanding chromosphere is sufficiently thick in dangerous frequencies. As soon as it develops, the inward flux of radiation checks the outflow, which starts again when the chromosphere, moving out as a detached shell, has dissipated itself into cosmic space. It appears plausible to ascribe the large velocities to the "forerunners," the order of a (§ 7) being in these cases not smaller than 3 or 6.

If the cycle is not very long (actually they are of the order of a few years) two and more "forerunners" can be observed simultaneously, thus ensuring the permanency of the total emission. At very large distances the "forerunners" can hardly move with a constant velocity, since the nebular L_α acting "from below" tends to accelerate them. On the other hand, dissipating into cosmic space, they should experience at large distances the above-mentioned "contour effect" in its full strength. Thus, after some cycles after the first explosion, the star can be expected to be embedded in the cloud of slowly moving atoms, gradually dissipating into cosmic space.

§ 9. *Bse Stars*

If the initial velocity (or more strictly μv_0) is small, a non-static spherically symmetrical chromosphere can exist only if τ_1 is small. In these cases the bright lines obviously should be both narrow and weak. Such is the emission in Bse stars, the representative being χ^2 Orionis. Merrill has shown (*op. cit.*) that H emission is much weaker in s-stars than in n-stars, the latter having a tendency to show intense lines or none at all. This corroborates our scheme and to some extent contradicts Struve's idea, that, roughly speaking, Bse stars are Bne stars observed in the direction of the axis of rotation. It goes without saying that some Bse stars probably fit Struve's scheme.

As a corollary to the above theory we may expect that the luminosity sequence in B stars is in the decreasing order : P Cygni, Bse, Bne and finally common absorption stars (as a matter of fact they comprise also some Be stars, which do not show bright lines except temporarily). This is in good agreement with all that we know about these stars, and indirectly with the well-known increase in luminosity from normal Bn to normal Bs stars.

B stars with *c*-characteristics, without or with faint emission lines of P Cygni type (*e.g.* Rigel), are intermediate in luminosity between P Cygni and Bse stars.

The contours of Bne emission and their variation will be investigated in a special paper.

Appendix on the Transmission of Radiation through a Moving Medium

In a moving nebular envelope the total net flux of radiation over a spherical surface in general cannot remain constant along the radius, as it happens to be in the static case if the sources and sinks of energy are absent. The radiation may drive the atoms away from the star, endowing them with positive acceleration, or if the envelope is thick enough, the inward stream of radiation may exert braking action on the outflowing atoms, starting from the star with a certain initial velocity. In both cases the radiation is doing work against or in the direction of the Newtonian field gradient; the net flux over a spherical surface should therefore decrease or increase with the increasing distance from the star. From a microscopic point of view the energy of radiation is being transformed into mechanical energy as a result of repeated atomic Compton effects for the ultra-violet quanta, producing macroscopic changes in both streams of radiation.

In order to estimate the change in total net flux we consider an idealized case, paying no attention to the partition of radiation amongst various frequencies. The complete equation of the conservation of energy in moving media is in Milne's form *

$$\frac{\partial E}{\partial t}+\rho\frac{D(C_v T)}{Dt}-\frac{p+p'}{\rho}\frac{D\rho}{Dt}=\epsilon\rho-\operatorname{div} F,$$

where $\frac{D}{Dt}$ means Lagrangian derivative, *i.e.* "following the element," E the density of radiation, p and p' the pressures, ϵ the rate of liberation of sub-atomic energy and F the net flux per unit surface. Applying the equation of continuity

$$\frac{D\rho}{Dt}=-\rho\operatorname{div} v,$$

where v is the velocity of flow (which is supposed here to be radial), to a steady isothermal case in the absence of sources of energy, neglecting hydrostatic pressure, we get

$$-p'\operatorname{div} v=\operatorname{div} F=\operatorname{div}\frac{C(r)}{r^2}, \qquad (1a)$$

where $C(r)$ is the net flux over a spherical surface r divided by 4π. To simplify the computations we assume $v=ar^n$, taking $n\leqslant 0$ in order to avoid infinite velocities. If R is the radius of star, the initial velocity is $v_*=aR_*^n$. On the other hand, for such a radially expanding envelope, starting from

* *Oxford Quarterly Journ. of Math.*, **1**, 8, 1930.

$r = R_*$ and extending to infinity, hydrodynamics give

$$\frac{dp'}{dr} = \rho\left(\frac{dV}{dr} - v\frac{dv}{dr}\right), \tag{2a}$$

where V is the Newtonian potential of the star, so that $\frac{dV}{dr} = -gR_*^2/r^2$ or g'/r^2, g being the surface gravity. Substituting the assumed expression for v, calculating p' from (1*a*), differentiating it and substituting into (2*a*), we get after the elimination of ρ by means of the equation of continuity $\rho r^2 v = \mu = \text{const.}$

$$\mu(n+2)[a^2 n r^{2n+1} + g'] = r^3\frac{d^2C}{dr^2} - (n+1)r^2\frac{dC}{dr};$$

substituting

$$\frac{dC}{dr} = P(r)$$

we get

$$\frac{dP}{dr} - \frac{(n+1)}{r}P = -\mu(n+2)[a^2 n r^{2n-2} + g'/r^3],$$

whence

$$P(r) = Ar^{n+1} - \mu(n+2)[g'/(n+3)r^2 - a^2 n r^{2n-1}/(n-2)],$$

and after a second integration

$$C(r) = Ar^{n+2}/(n+2) + \mu(n+2)[g'/(n+3)r + a^2 r^{2n}/2(n-2)] + B,$$

A and B being the constants of integration, values $n = -2$ and $n = 0$, corresponding to the logarithmic cases, being excluded. For the sake of simplicity we consider the values $0 > n > -2$ when for finite C the constant A is zero.

In these cases $B = C(\infty)$ and

$$C(R_*) - C(\infty) = \mu(n+2)[gR_*/(n+3) + v_*^2/2(n-2)] = \tfrac{1}{2}\mu(n+2)\left[\frac{v_p^2}{n+3} + \frac{v_*^2}{n-2}\right], \tag{3a}$$

where v_p is the parabolic velocity at the stellar surface. We infer that for $n = -\frac{1}{2}$ and $v_* = v_p$ the total net flux is constant throughout the envelope, as it should be since this case corresponds to simple gravitational escape when the radiation does not perform any work. For other values of n, the initial and final fluxes are equal when $v_* = v_p\sqrt{(2-n)}/(n+3)$. If v_* is smaller than this critical value, $C(R_*) - C(\infty)$ is positive, otherwise it is negative, as should be expected from elementary considerations. In general

$$C(r) = C(R_*) + \frac{\mu(n+2)}{2}\left[\frac{v_p^2}{n+3}\left(\frac{R_*}{r} - 1\right) + \frac{v_*^2}{n-2}\left(\frac{r^{2n}}{R_*^{2n}} - 1\right)\right],$$

$C(r)$ is extremal at $r' = R_*\left[\frac{v_p^2(n-2)}{v_*^2 2n(n+3)}\right]^{\frac{1}{2n+1}}$, which corresponds to a minimum if $n > -\frac{1}{2}$ and to a maximum if $n < -\frac{1}{2}$, while $n = -\frac{1}{2}$ corresponds to a monotonic increase or decrease in C. Whether the extremal value of $C(r)$ is only formal ($r' < R_*$) or has a physical significance ($r' > R$) depends upon v_*.

In order that the approximation $C = \text{const.}$ could be used in investigating the field of radiation in the envelope it is necessary that, say, $\left|\frac{C(r') - C(R_*)}{C(R_*)}\right| < 0{\cdot}01$. At present it seems difficult to prove that this condition is really satisfied in Be stars. We can only show that this condition is not contradicted by what we know about these stars.

Since in these cases the assumed value of n makes no difference we take $n = -\frac{1}{2}$ when $C(r)$ changes monotonically. Then

$$C(R_*) - C(\infty) = 0{\cdot}3\mu(v_p^2 - v_*^2).$$

The approximation $\left|\frac{C(R_*) - C(\infty)}{C(\infty)}\right| < 10^{-2}$ is valid if $4\pi\mu < L/30|v_p^2 - v_*^2|$ where L is the total radiation. As an extreme case take the P Cygni model. Here $L = 1{\cdot}2 \times 10^6 L\odot$, $R_* = 100 R\odot$, $g = 1{\cdot}3{\cdot}10^2$, $v_p^2 = 1{\cdot}8 \times 10^{15}$, $v^2 = 4{\cdot}8{\cdot}10^{14}$, and the above condition can be written $4\pi\mu < 1{\cdot}2 \times 10^{23}$ gr. The displacement of the absorption line is 220 km./sec.; moreover, for such a distended star p_e at the absorption level can hardly be larger than, say, 10^{-6} atm., which for $T = 20{,}000°$ corresponds to $\rho = 6{\cdot}10^{-13}$. For assumed p_{e*} this gives $4\pi\mu = 8{\cdot}10^{21}$ and $8{\cdot}10^{20}$ for $p_{e*} = 10^{-7}$, thus making valid the approximation $C(r) = \text{const.}$ for the total flux.

For our purposes more important, however, is the flux in the continuous spectrum beyond the Lyman limit, and it seems of interest to determine the order of error introduced by the approximation in this case without taking into account the subsequent redistribution of C radiation amongst various frequencies (except that of L_α). Considering the nebular region we take in the denominator $C(R_0)$ instead of $C(\infty)$, R_0 being the radius of the lower boundary. Here v_p can be neglected, while v_0 is of the order of 150 km./sec. The equilibrium intensity of C radiation at $T = 20{,}000°$ is $1{\cdot}1 \times 10^{-2}$ smaller than that of the total radiation. Repeating the above arguments we find that the error of approximation is smaller than 1 per cent. if $4\pi\mu < 8{\cdot}10^{21} q$, where q is the reduction caused by the absorption in the lower strata. If $q \geqslant 0{\cdot}1$, which appears to be a not unreasonable guess (otherwise the application of Zanstra's method to such stars could not be so successful), the error of approximation is of the order of 10 per cent. or 1 per cent. according as $p_{e*} = 10^{-6}$ or 10^{-7}.

We have purposely considered P Cygni stars which exhibit especially violent outflow; however, even here the loss of mass does not seem prohibitive for the approximation $C(r) = \text{const.}$

Having in view all the uncertainty in the numerical data we may say that the above calculations do not reveal any strong argument against our using the approximation $C(r) = \text{const.}$ when investigating the radiation field in an expanding nebular envelope.

We pass to the equation of radiative transfer in moving media. To make clear all the conditions of the problem we shall consider it hydrodynamically, dealing with a cylindrical case. Let dS be the elementary plane surface, moving with constant velocity v along its normal with respect to a certain reference frame. Consider a fluid flowing with constant velocity

u in the direction making angle θ with the normal to dS within an elementary solid angle $d\omega$. Let $i(\theta)\cos(\theta)dSd\omega$ be the amount of fluid crossing dS per second if the surface were at rest. Put another dS identical to the first at a distance $ds=(u-v\cos\theta)dt$ in the direction of the flow, dt being the time necessary for the fluid to pass the distance ds. It is easy to see that the increase in flow per second for ds along the line of flow is

$$\frac{Di(\theta)}{Ds}\cos\theta\, dSd\omega dt=\left[\frac{\partial i(\theta)}{\partial t}+(u-v\cos\theta)\frac{\partial i(\theta)}{\partial s}\right]\cos\theta dSd\omega dt, \qquad (4a)$$

since $\frac{D}{Dt}$ is the Lagrangian derivative. Let $\Sigma e d\omega$ be the combined intensity of "sources" and "sinks" for $i(\theta)$ within $d\omega$ per second and unit volume. The net gain in flow per second in cylindrical volume $dsdS\cos\theta$ is

$$(u-v\cos\theta)\cos\theta\,\Sigma e\,.\,dtd\omega dS. \qquad (5a)$$

Equating (4a) and (5a) we get

$$\frac{\partial i(\theta)}{\partial t}+(u-v\cos\theta)\frac{\partial i(\theta)}{\partial s}=(u-v\cos\theta)\Sigma e,$$

which is the equation of fluid transfer in moving space endowed with sources and sinks.

Considering the radiation as a fluid moving with the velocity of light we cannot omit the following complication. The velocity u entering the equation of transfer is the velocity of the transfer of amplitude, which in non-dispersive media, or in dispersive media for ν far from the resonance frequencies, is equal to the "signal velocity" of the light. In these cases, by Einstein's composition theorem, $u-v\cos\theta=u=c$. On the other hand, in dispersive media close to the resonance frequencies u is not the "signal velocity," but has a smaller and rapidly varying value. Here Einstein's theorem is not applicable. For a definite frequency ν put

$$\Sigma e_\nu(\theta)=-\kappa_\nu\rho i_\nu(\theta)+E_\nu,$$

E_ν being the emission per unit solid angle, κ_ν and ρ absorption coefficient and density. Then for ν far from the resonance frequencies we get for this cylindrical problem

$$\frac{1}{c}\frac{\partial i_\nu(\theta)}{\partial t}+\frac{\partial i_\nu(\theta)}{\partial s}=-\kappa_\nu\rho i_\nu(\theta)+E_\nu, \qquad (6a)$$

and for ν sufficiently close to the resonance frequencies

$$\frac{1}{u}\left[1+\frac{v}{u}\cos\theta\right]\frac{\partial i_\nu(\theta)}{\partial t}+\frac{\partial i_\nu(\theta)}{\partial s}=-\kappa_\nu\rho i_\nu(\theta)+E_\nu,$$

u being a function of ν.

The formula (6a) was derived by Rosseland.*

* *Ap. J.*, 63, 342, 1926.

For a steady state both formulæ degenerate into

$$\frac{\partial i_\nu(\theta)}{\partial s} = -\kappa_\nu \rho i_\nu(\theta) + E_\nu. \qquad (7a)$$

It is necessary to observe that κ_ν and E_ν are known in the frame of reference at rest with respect to the absorbing and emitting atoms; thus, if transformed to the standard frame of reference, this equation should contain the terms in $\frac{v}{c}$. On the other hand, as the optical thickness, τ, is calculated with the value of κ_ν referred to the atoms at rest, the equation of transfer, if transformed to τ as variable, remains the same as in the static case correct to the first order of $\frac{v}{c}$. In such a manner, investigating the transfer of radiation through an expanding nebula in a steady state, we may use the equation corresponding to the static case.

Poulkovo:
1934 *January*.

OBSERVATIONS OF NOVÆ, 1933–1934.

W. H. Steavenson.

The observations which follow were all made with the 20-inch Hindle reflector. Of the nine novæ dealt with in the paper, six have never shown any great variation in light since falling to their minimum, and I have therefore not thought it worth while to observe them very frequently. The remaining three have exhibited conspicuous variability within recent years, and have consequently been kept fairly regularly under observation. Even with these there have occurred considerable periods of time during which little variation was shown, as in the case of Nova Ophiuchi (1848) in 1933; but I have felt justified in recording the considerable number of very similar estimates here given in view of the past history of the nova, believing that such a period of quiescence may be only temporary and may be as significant as the variability previously exhibited.

References to the comparison stars used were published in *M.N.*, **92**, 720.

Nova Aquilæ (1918)

Date	Mag.	Date	Mag.
1933 May 19	10·6	1933 July 2	10·7
27	10·7	20	10·8
June 1	10·65	23	10·6
18	10·7	Aug. 23	10·8
28	10·7		

For some years now the visual magnitude of this nova has undergone no changes at all comparable in range with the variations said to have been shown on photographs taken before the outburst of 1918. The image is sensibly stellar in appearance, and I can see no nebulosity surrounding it.

Nova Aurigæ (1892)

Observed on five nights. The image is quite stellar and free from nebulosity. I believe the slight variability indicated by the observations to be real, and I have observed similar small changes in previous years.

Date	Mag.
1933 Oct. 16	14·8
19	14·7
Nov. 19	14·9
1934 Mar. 5	14·7
18	14·9

Nova (T) Coronæ (1866)

Observed on two nights. This nova is unlike the others in being markedly yellow in colour. It has a stellar image, free from nebulosity, and appears to be quite steady in light.

Date	Mag.
1933 June 28	9·96
Aug. 27	9·96

Nova Cygni (1920)

Observed on two nights. The general fading appears now to have ceased, and there is little or no variability.

Date	Mag.
1933 June 21	15·5
Nov. 19	15·4

Nova Cygni (1876)

Observed on forty-one nights. This is one of the novæ which are undoubtedly variable. The changes in 1933 were smaller than usual, the total range amounting to only $0^{m}.41$, and there is no strong indication of any definite period. The image is stellar and free from nebulosity. It may be of interest to note that the mean magnitude is almost exactly equal to that of the planet Pluto.

Date	Mag.	Date	Mag.	Date	Mag.
1933 June 1	15·00	1933 July 23	14·86	1933 Aug. 24	14·76
14	14·81	24	14·81	25	14·81
18	14·86	25	14·81	26	14·81
19	14·86	26	14·86	27	14·76
21	14·91	27	14·81	28	14·81
28	14·91	31	14·86	29	14·76
July 2	14·76	Aug. 1	14·86	Sept. 9	14·71
3	14·81	8	14·86	10	14·71
4	15·01	13	14·86	21	14·71
7	14·86	14	14·86	Oct. 19	14·60
12	14·96	16	14·81	25	14·91
14	14·76	18	14·86	Nov. 15	14·91
18	14·81	20	14·91	19	14·91
19	14·86	23	14·81		

Nova Geminorum (1912)

Observed on one night only, 1933 November 19, when the magnitude was estimated to be 14·5. No sensible variation in the light of this object has been noted for some years. The faint nebulous glow seen round the nova with the 28-inch refractor of the Royal Observatory does not appear in the reflector, and is probably of optical origin. The image itself is stellar.

Nova Lacertæ (1910)

Observed on two nights. This nova reached a stationary condition many years ago and has since exhibited no definite variability. It is stellar and free from nebulosity.

Date	Mag.
1933 June 19	14·3
Aug. 23	14·4

Nova Ophiuchi (1848)

Observed on thirty-six nights. The variations have been small, amounting to only $0^m.4$ during the period. The image is perhaps not quite perfectly stellar, but there is no surrounding nebulosity.

Date	Mag.	Date	Mag.	Date	Mag.
1933 May 19	12·7	1933 June 21	12·6	1933 July 27	12·7
21	12·8	28	12·8	29	12·7
22	12·8	30	12·8	30	12·7
27	12·8	July 2	12·7	Aug. 1	12·7
June 1	12·7	3	12·7	8	12·6
3	12·7	7	12·7	16	12·7
4	12·6	12	12·6	18	12·5
8	12·6	18	12·6	23	12·4
14	12·8	19	12·5	25	12·7
17	12·7	24	12·7	26	12·7
18	12·8	25	12·5	27	12·7
19	12·7	26	12·6	28	12·7

Nova Persei (1901)

Observed on ninety-six nights. This is the most conspicuously variable of all the novæ. The observations which follow show a range of $1^m.05$, which is rather smaller than usual. The changes of light are for the most part sudden and irregular, and the figures can scarcely be said to indicate any definite period. The wisp of nebulosity attached to the nova was well seen on the more transparent nights, but I have the impression that it is fainter now than when I first observed it, in 1916 September. The image itself is stellar.

[Table

Date	Mag.	Date	Mag.	Date	Mag.
1933 July 24	13·77	1933 Oct. 17	13·77	1934 Jan. 26	13·77
25	13·52	18	13·82	Feb. 1	13·82
26	13·57	19	13·77	2	13·36
Aug. 1	12·92	25	13·62	6	13·26
3	13·10	26	13·57	7	13·25
4	13·10	27	13·82	8	13·00
7	13·05	Nov. 1	13·72	9	13·42
8	13·25	8	13·62	10	13·42
12	13·41	9	13·67	11	13·82
14	13·52	12	13·41	12	13·57
16	13·47	13	13·36	13	13·87
18	13·47	15	13·36	18	13·62
20	13·47	19	13·57	19	13·77
21	13·42	Dec. 3	13·57	21	13·77
23	13·21	4	13·52	22	13·57
24	12·90	8	13·41	Mar. 3	13·42
25	12·95	9	13·52	5	13·47
26	13·00	12	13·47	6	13·42
27	12·82	13	13·47	7	13·30
28	13·10	14	13·47	8	13·10
30	13·31	15	13·42	10	12·84
Sept. 2	13·57	17	13·47	11	13·36
4	13·67	1934 Jan. 1	13·87	16	13·31
5	13·87	3	13·72	17	13·36
6	13·87	5	13·42	18	12·87
9	13·57	8	13·52	27	13·82
10	13·57	9	13·36	30	13·25
14	13·77	12	12·95	31	13·00
15	13·72	15	13·31	Apr. 12	13·36
21	13·67	18	13·36	18	12·84
Oct. 12	13·82	19	13·42	19	13·10
16	13·77	22	13·20	20	13·20

THE STABILITY OF GASEOUS STARS

T. G. Cowling, M.A., D.Phil.

1. The problem of the stability of gaseous stars has been studied by various authors, beginning with Jeans.* There are three classes of criteria of stability.

First, there are the dynamical criteria, expressing the condition that no approximately adiabatic motion can be set up and maintained which is not oscillatory. For dynamical stability of the whole star it is necessary that γ, the ratio of the specific heats of the gas, shall exceed $4/3$; this is a necessary condition of existence of the star. Also, if convection currents are not to be

* *M.N.*, **85**, 914, 1925 ; **87**, 400, 720, 1927. An inaccuracy in Jeans's work arising from an incorrect equation of energy was corrected by Vogt, *Ast. Nach.*, 232, No. 5545, 1928. In the limiting case quoted in the text no correction is necessary.

set up in any layer, the radiative temperature gradient must be less than the corresponding adiabatic gradient.

Next, there is the vibrational condition of stability, which secures that in any approximately adiabatic oscillation the combined effect of the generation of energy, the flux of radiation, etc., will be to decrease and not to increase the amplitude of the motion. Jeans obtained such a condition, supposing that a radial vibration would occur in which the outward displacement of any spherical shell of the gas concentric with the star is proportional to its radius. If ϵ, the rate of generation of energy per unit mass, and κ, the coefficient of opacity, are connected with the density ρ and the temperature T by relations of the form

$$\epsilon = \epsilon_0 \rho^\alpha T^\beta, \qquad \kappa = \kappa_0 \rho^n / T^s \tag{1}$$

then for a model in which the ratio of radiation to gas pressure is small, and the ratio of the specific heats is γ, the condition for stability found by Jeans's method is

$$\alpha + \beta(\gamma - 1) \leqslant \tfrac{4}{3}(3\gamma - 4) + s(\gamma - 1) - n. \tag{2}$$

As, however, the assumption that the displacement of any mass of gas is proportional to its distance from the centre can only be approximately true,* this criterion can only be approximately valid.

Lastly, the model must possess thermal stability, that is, in a disturbed state of smaller radius approximately in mechanical equilibrium the resultant effect of generation of energy and flow of heat must be to heat up the star and so cause it to expand ; and conversely for a disturbed state of larger radius. The condition for this obtained under the same assumptions as were used to derive (2) is

$$3\alpha + \beta \geqslant s - 3n. \tag{3}$$

This condition is, however, more accurate than (2). If the inequality in this relation is replaced by equality there is neutral equilibrium, since a neighbouring state of equilibrium exists such that the displacement of any mass of gas is proportional to its distance from the centre ; thus if the expressions on the two sides of (3) are nearly equal there will exist an almost neutral neighbouring state for which the displacement of any mass is almost proportional to its distance from the centre. Hence if $3\alpha + \beta - s + 3n$ is not too large, the condition for thermal stability is accurately (3). We therefore need only concern ourselves with the other conditions of stability.

2. Several authors † have pointed out that if the rate of generation of energy per unit mass increases rapidly with the temperature the radiative temperature gradient near the centre of the star may exceed the corresponding adiabatic gradient, when convection will be set up. For, consider the point-source model. This possesses two types of configuration—the centrally condensed type with infinite central density and temperature, and the

* Compare, *e.g.*, Eddington's work on Cepheid variation (*M.N.*, **79**, 2, 177, 1919) and L. H. Thomas's discussion of the slow contraction of a star (*M.N.*, **91**, 122, 1930).

† See, *e.g.*, Rosseland, *Zeit. für Astrophys.*, 4, 255, 1932 ; Biermann, *Zeit. für Astrophys.*, **5**, 117, 1932.

collapsed type with finite temperature and zero density at the centre. These can each be regarded as limiting cases of models for which the rate of generation varies as $(T/T_0)^\beta$, when in the first case T_0 and in the second β becomes infinite. Thus if the rate of generation varies very rapidly with temperature the configurations will approximate closely to the collapsed configurations of the point-source model, so that density will decrease as the centre is approached, and the star will be convectively unstable.

In general, the effect of a concentration of sources towards the centre is that radiation pressure is a larger fraction of the total pressure near the centre than in the outer layers, and varies more rapidly in proportion to its magnitude than the total pressure near the centre. If, then, the radiation pressure is a fairly large fraction of the total pressure near the centre, only a small part of the mass will be supported against gravity by the gas pressure, and so the density gradient will be small. Thus the adiabatic temperature gradient will be small, and convective instability is accordingly probable. The fact that the effective ratio of specific heats for a mixture of gas and radiation is lower than that for gas alone operates the same way. For a given law of generation of energy, stability is most likely to obtain in the limiting case when radiation pressure is negligible compared with gas pressure and the ratio of specific heats of the material has its maximum value 5/3.

3. To obtain some idea as to when a star is convectively unstable, numerical integration has been effected for the cases in which $\epsilon \propto T^\beta$ and either $\kappa = \text{const.}$ or $\kappa \propto \rho/T^{7/2}$. Suppose for generality that κ, ϵ are given by equations (1). Then if radiation pressure is negligible compared with gas pressure, the equations governing the problem are

$$\frac{d}{dr}\left(\frac{R}{\mu}\rho T\right) = -\frac{Gm\rho}{r^2}, \qquad \frac{d}{dr}(\tfrac{1}{3}aT^4) = -\frac{\kappa_0 l\rho^{1+n}}{4\pi c r^2 T^s},$$

where m, l denote the mass inside a sphere of radius r concentric with the star and the rate at which this mass generates energy; μ is the molecular weight, R the gas constant, a the radiation constant, G the constant of gravitation, c the velocity of light. We have also

$$\frac{dm}{dr} = 4\pi\rho r^2, \qquad \frac{dl}{dr} = 4\pi\rho r^2 \,.\, \epsilon_0\rho^\alpha T^\beta.$$

In these equations set

$$T = T_0\tau, \quad \rho = \rho_0\sigma, \quad r = r_0 x, \quad m = m_0\mu, \quad l = l_0\lambda,$$

where T_0, ρ_0 are the central temperature and density, and r_0, m_0, l_0 are quantities of the dimensions of a length, a mass, and a luminosity. Then, following Milne * and Rosseland,† if r_0, m_0, l_0 are suitably chosen, the equations reduce to

$$\frac{d}{dx}(\sigma\tau) = -\frac{K\mu\sigma}{x^2}, \qquad \frac{d\tau}{dx} = -\frac{\lambda\sigma^{1+n}}{\tau^{3+s}x^2},$$

$$\frac{d\mu}{dx} = \sigma x^2, \qquad \frac{d\lambda}{dx} = \rho^\alpha\sigma^{1+\beta}x^2,$$

* *Zeit. für Astrophys.*, 4, 75, 1932. † *Ibid.*, 4, 255, 1932.

where K is a dimensionless constant. The four constants appearing in the general solution of these equations are fixed by the fact that $\sigma = \tau = 1$, $\lambda = \mu = 0$ when $x = 0$.

The equations are solved by an outward numerical integration. By varying K a single infinity of solutions can be obtained; for only one value K_0 of K do gas and radiation pressures vanish simultaneously at a finite boundary. If $K > K_0$, gas pressure vanishes first, if $K < K_0$, radiation pressure: the case $K = K_0$ corresponds closely to physical possibility. It is, however, not necessary for the present purpose to find K_0 for each model. For convective stability

$$\frac{1}{\sigma\tau}\frac{d(\sigma\tau)}{dx}\Big/\frac{1}{\tau}\frac{d\tau}{dx} \geqslant \frac{\gamma}{\gamma - 1} \geqslant \frac{5}{2}$$

and, for the models we have to consider, the quantity on the left of this relation attains its maximum value at the centre. Assume, then, that it is just equal to 5/2 at the centre, so that $K = 5/2$. If, now, radiation pressure is found to vanish before gas pressure, no convectively stable model exists; if the converse, such models can exist.

The results obtained are as follows:—

(i) If κ is constant and $\epsilon \propto T^{\beta}$, no convectively stable model is possible if β exceeds a number lying between 3 and 4.

(ii) If $\kappa \propto \rho/T^{7/2}$ and $\epsilon \propto T^{\beta}$, no convectively stable model is possible if β exceeds a number lying between 7 and 8. The difference between the results for this case and the case $\kappa =$ constant is noteworthy; it is, in fact, only true that the properties of models constructed in accordance with these two opacity laws are quantitatively similar if ϵ is nearly constant.

4. We now turn to the problem of vibrational stability. Rosseland * has studied this problem, using a perturbation method similar to that used in atomic theory. He derives a criterion which requires a determination of the possible adiabatic oscillations for its exact application to any special model. The general order of the results may, however, be obtained without such detailed computation.

As a preliminary we derive Rosseland's result in a slightly modified form. Consider the equations governing a radial oscillation of a star which is convectively stable. All the variables will be regarded as functions of the variable m of § 3 and of the time t.† Then

$$\frac{\partial(\frac{4}{3}\pi r^3)}{\partial m} = \frac{1}{\rho}. \tag{4}$$

If p is the total pressure, the equation of motion is

$$\frac{\partial^2 r}{\partial t^2} = -4\pi r^2 \frac{\partial p}{\partial m} - \frac{Gm}{r^2}; \tag{5}$$

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* *Pub. Obs. Oslo*, Nos. 1, 2, 1931. The essential parts of Rosseland's work are set out in a paper by Edgar, *M.N.*, 93, 422, 1933.

† Use of these variables was made by Thomas (*loc. cit.*).

and if U is the energy enclosed in unit volume, the equation of energy is

$$\frac{\partial}{\partial t}\left(\frac{U}{\rho}\right)+p\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)=\epsilon-\frac{\partial F}{\partial m}, \tag{6}$$

where

$$F=-\frac{16\pi^2cr^4}{\kappa}\frac{\partial(\frac{1}{3}aT^4)}{\partial m}=-\frac{16\pi^2cr^4}{\kappa}\frac{\partial p'}{\partial m} \tag{7}$$

and p' is the radiation pressure. In an equilibrium state, (5) and (6) reduce to

$$\frac{\partial p}{\partial m}=-\frac{Gm}{4\pi r^4}, \qquad F=\int_0^m \epsilon dm=l. \tag{8}$$

In a state slightly disturbed from equilibrium we set $r=r_0+\delta r$, etc., where r_0, etc., are the equilibrium values, and squares and products of δr, $\delta\rho$, etc., are neglected. Then we find that

$$\frac{\partial(4\pi r_0^2\delta r)}{\partial m}=-\frac{\delta\rho}{{\rho_0}^2}$$

$$\frac{\partial^2(\delta r)}{\partial t^2}=-4\pi {r_0}^2\frac{\partial\delta p}{\partial m}+\frac{4Gm\delta r}{{r_0}^3}$$

$$\frac{1}{\rho_0}\frac{\partial(\delta U)}{\partial t}-\frac{(U_0+p_0)}{{\rho_0}^2}\frac{\partial\delta\rho}{\partial t}=\delta\epsilon-\frac{\partial\delta F}{\partial m}.$$

In subsequent work we omit the suffix from the equilibrium values.

The boundary conditions are taken to be that p, δp vanish at the boundary, where we set $m=M$, $l=L$, $r=R$. For a normal oscillation δr, etc., may be supposed each to involve the time through a factor $e^{\iota\sigma t}$ if in interpreting our results we take the real part of all complex quantities. Then the above equations become

$$\frac{\partial(4\pi r^2\delta r)}{\partial m}=-\frac{\delta\rho}{\rho^2}, \qquad \left(\sigma^2+\frac{4Gm}{r^3}\right)\delta r=4\pi r^2\frac{\partial\delta p}{\partial m}, \tag{9}$$

$$\frac{\delta U}{\rho}-\frac{(U+p)\delta\rho}{\rho^2}=-\frac{\iota}{\sigma}\left(\delta\epsilon-\frac{\partial\delta F}{\partial m}\right). \tag{10}$$

As the last equation involves ι, σ is in general complex. Since, however, the terms on the left-hand side of this equation are large compared with those on the right except in a shallow layer near the boundary, the values of σ will differ but little from those corresponding to adiabatic oscillations.

Consider such oscillations therefore. For these (10) is replaced by

$$\frac{\delta U}{\rho}-\frac{(U+p)\delta\rho}{\rho^2}=0.$$

Now δU is a linear function of δp, $\delta\rho$, say

$$\delta U=A\delta p+B\delta\rho. \tag{11}$$

Thus

$$\delta p = (U + p - B\rho)\delta\rho / A\rho = C\delta\rho/\rho, \tag{12}$$

say. Hence, by (9),

$$\left(\sigma^2 + \frac{4Gm}{r^3}\right)\delta r = -16\pi^2 r^2 \frac{\partial}{\partial m}\left(C\rho \frac{\partial}{\partial m}(r^2 \delta r)\right). \tag{13}$$

We require solutions of this equation which give $\delta r = 0$ if $m = 0$ and $\delta p = 0$ if $m = M$. These conditions can only be satisfied for certain values σ_1, σ_2, . . . of σ. Let $\delta r_1 e^{\iota\sigma_1 t}$, $\delta r_2 e^{\iota\sigma_2 t}$, . . . be the corresponding values of δr, $\delta\rho_1 e^{\iota\sigma_1 t}$, $\delta\rho_2 e^{\iota\sigma_2 t}$, . . . those of $\delta\rho$, etc. Then δr_k satisfies (13), with $\sigma = \sigma_k$; thus by the Sturm-Liouville theorem, if $k \neq l$,

$$\int_0^M \delta r_k \delta r_l \, dm = 0 \tag{14}$$

and the functions δr_k form a complete orthogonal set, so that any other function of m can be expanded in a series of them.

Now return to the general case, when the oscillations are not adiabatic. Let

$$\sigma = \sigma_k + \sigma_k', \qquad \delta r = (\delta r_k + \delta r_k')e^{\iota\sigma t}, \tag{15}$$

where σ_k', $\delta r_k'$ are small, give the oscillation of period nearly equal to $2\pi/\sigma_k$. For this oscillation (12) is replaced by

$$\delta p = \frac{C\delta\rho}{\rho} - \frac{\iota\rho}{A\sigma}\left(\delta\epsilon - \frac{\partial \delta F}{\partial m}\right),$$

or, as the terms involving $\delta\epsilon$, δF are small, to a first approximation

$$\delta p = \frac{C\delta\rho}{\rho} - \frac{\iota\rho e^{\iota\sigma t}}{A\sigma_k}\left(\delta\epsilon_k - \frac{\partial \delta F_k}{\partial m}\right).$$

Hence (13) is replaced by

$$\left(\sigma^2 + \frac{4Gm}{r^3}\right)\delta r = -4\pi r^2 \frac{\partial}{\partial m}\left(C\rho \frac{\partial}{\partial m}(4\pi r^2 \delta r) + \frac{\iota\rho e^{\iota\sigma t}}{A\sigma_k}\left(\delta\epsilon_k - \frac{\partial \delta F_k}{\partial m}\right)\right),$$

or, substituting from (15) and using the equation for δr_k, and neglecting products of small quantities,

$$\left(\sigma_k^2 + \frac{4Gm}{r^3}\right)\delta r_k' + 2\sigma_k \sigma_k' \delta r = -4\pi r^2 \frac{d}{dm}\left(C\rho \frac{d}{dm}(4\pi r^2 \delta r_k') + \frac{\iota\rho}{A\sigma_k}\left(\delta\epsilon_k - \frac{d\delta F_k}{dm}\right)\right).$$

Put

$$\delta r_k' = \sum_l A_{kl} \delta r_l.$$

Then, on use of the equation for δr_l,

$$\sum_l A_{kl}(\sigma_k^2 - \sigma_l^2)\delta r_l + 2\sigma_k \sigma_k' \delta r_k = -\frac{4\pi\iota r^2}{\sigma_k} \frac{d}{dm}\left(\frac{\rho}{A}\left(\delta\epsilon_k - \frac{d\delta F_k}{dm}\right)\right).$$

Multiply by δr_k and integrate with respect to m over the whole mass of the

star. Then, by (14),

$$2\sigma_k^2\sigma_k'\int_0^M(\delta r_k)^2dm = -\iota\int_0^M 4\pi r^2\delta r_k\frac{d}{dm}\left(\frac{\rho}{A}\left(\delta\epsilon_k-\frac{d\delta F_k}{dm}\right)\right)dm$$
$$= -\iota\int_0^M\frac{\delta\rho_k}{A\rho}\left(\delta\epsilon_k-\frac{d\delta F_k}{dm}\right)dm$$

on integrating by parts and using (9). If the amplitude of the oscillations is not to increase indefinitely, the imaginary part of σ must not be negative. Hence, dropping the suffix k, the condition of stability is that for every normal oscillation

$$\int_0^M\frac{\delta\rho}{A\rho}\left(\delta\epsilon-\frac{d\delta F}{dm}\right)dm\leqslant 0. \tag{16}$$

Roughly speaking, this implies that when any portion of the gas is compressed it must cool, and when rarified it must heat up. Since an increase in temperature accompanies an increase in density, the flow of energy from point to point may be expected to produce this result, especially for the oscillations of high frequency with many nodes. The effect of generation of energy will be in the reverse direction.

It was assumed in the derivation of (16) that changes in the temperature of the gas are approximately adiabatic. In the extreme outer layers of the star this is not so, and in consequence the effect of these layers on the stability is probably overestimated in (16). Since the region in which the adiabatic approximation breaks down is not very deep, however, the resulting error will be comparatively small, and we therefore neglect it.

5. For the present we shall consider a model such that the laws governing an adiabatic change are identical with the ordinary gas laws

$$\frac{\delta p}{p}=\frac{\gamma\delta\rho}{\rho},\qquad \frac{\delta T}{T}=\frac{(\gamma-1)\delta\rho}{\rho}, \tag{17}$$

where γ is a constant. The quantity A of equation (11) is then equal to $1/(\gamma-1)$. Equation (17) may be taken as approximately valid if radiation pressure is small compared with gas pressure, and, less closely, if the ratio of radiation to gas pressure, though not small, is roughly constant.

The laws of generation of energy and of opacity we suppose to be given by (1). Then, using (17),

$$\frac{\delta\epsilon}{\epsilon}=(\alpha+\beta(\gamma-1))\frac{\delta\rho}{\rho},\qquad \frac{\delta\kappa}{\kappa}=(n-s(\gamma-1))\frac{\delta\rho}{\rho}.$$

Hence, from (7),

$$\delta F=F\left(4\frac{\delta r}{r}-(n-s(\gamma-1))\frac{\delta\rho}{\rho}\right)-\frac{16\pi^2cr^4}{\kappa}\frac{d\delta p'}{dm}\ *$$

* Suppose that δr satisfies the relation $\delta r/r=\text{const.}=a$ instead of the second of equations (9). Then $\delta\rho/\rho=-3a$, $\delta T/T=-3(\gamma-1)a$, and so this relation can be reduced to

$$\delta F=F\frac{\delta\rho}{\rho}(-\tfrac{4}{3}-n+s(\gamma-1)+4(\gamma-1)).$$

or, using (8) and (9),

$$\delta F = \frac{l}{mG}\left(4\pi r^4 \frac{d\delta p}{dm} - \sigma^2 r^2 \delta r\right) - l(n - s(\gamma - 1))\frac{\delta\rho}{\rho} - \frac{16\pi^2 cr^4}{\kappa}\frac{d\delta p'}{dm}.$$

In order to apply the condition of stability (16), we need to evaluate the integral

$$\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d\delta F}{dm}\right)dm.$$

Now we have

$$\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(l\frac{\delta\rho}{\rho}\right)dm = \tfrac{1}{2}\int_0^M \left\{\frac{d}{dm}\left(l\left(\frac{\delta\rho}{\rho}\right)^2\right) + \epsilon\left(\frac{\delta\rho}{\rho}\right)^2\right\}dm$$

$$= \tfrac{1}{2}L\left(\frac{\delta\rho}{\rho}\right)^2_M + \tfrac{1}{2}\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

where a suffix M indicates that the quantity is evaluated at the boundary, given by $m = M$. Also

$$\frac{1}{\gamma}\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(4\pi r^4 \frac{l}{m}\frac{d\delta p}{dm}\right)dm$$

$$= \frac{1}{\gamma}\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left[4\pi r^4 \frac{l}{m}\left(p\frac{d}{dm}\left(\frac{\delta p}{p}\right) - \frac{\delta p}{p}\frac{dp}{dm}\right)\right]dm$$

$$= \int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(4\pi r^4 p\frac{l}{m}\frac{d}{dm}\left(\frac{\delta\rho}{\rho}\right) + Gl\frac{\delta\rho}{\rho}\right)dm$$

$$= -\int_0^M 4\pi r^4 p\left[\frac{d}{dm}\left(\frac{\delta\rho}{\rho}\right)\right]^2 dm + \tfrac{1}{2}GL\left(\frac{\delta\rho}{\rho}\right)^2_M + \tfrac{1}{2}G\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm$$

using an integration by parts, the integrated part vanishing at both limits. Again, by a similar transformation,

$$\frac{1}{4(\gamma - 1)}\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{16\pi^2 cr^4}{\kappa}\frac{d\delta p'}{dm}\right)dm$$

$$= \int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{16\pi^2 cr^4 p'}{\kappa}\frac{d}{dm}\left(\frac{\delta\rho}{\rho}\right) + l\frac{\delta\rho}{\rho}\right)dm$$

$$= -\int_0^M \frac{16\pi^2 cr^4 p'}{\kappa}\left[\frac{d}{dm}\left(\frac{\delta\rho}{\rho}\right)\right]^2 dm + \tfrac{1}{2}L\left(\frac{\delta\rho}{\rho}\right)^2_M + \tfrac{1}{2}\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

Thus

$$\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d\delta F}{dm}\right)dm = (\alpha + n + (\beta - s)(\gamma - 1) - \tfrac{4}{3}(3\gamma - 4))\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

and so condition (16) reduces to (2), as is to be expected. Hence, if the assumption $\delta r/r = \text{const.}$ is used in conjunction with Rosseland's criterion (as was done by Steensholt, *Zeit. für Astrophys.*, **7**, 373, 1933; **8**, 56, 1934) the results are not necessarily more accurate than those given by Jeans's condition.

Using these results, we obtain the relation

$$\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d\delta F}{dm}\right)dm = (\alpha+\beta(\gamma-1)-\tfrac{1}{2}(s(\gamma-1)-n+3\gamma-4))\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm$$
$$-\tfrac{1}{2}(s(\gamma-1)-n+3\gamma-4)L\left(\frac{\delta\rho}{\rho}\right)_M^2 + \int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{l\sigma^2 r^2\delta r}{Gm}\right)dm$$
$$-\int_0^M 4\pi r^4\left(\frac{16\pi(\gamma-1)cp'}{\kappa} - \frac{\gamma pl}{Gm}\right)\left[\frac{d}{dm}\left(\frac{\delta\rho}{\rho}\right)\right]^2 dm, \qquad (18)$$

and for stability this is to be negative.

Consider, *e.g.*, a model for which ϵ is constant, so that $\alpha=\beta=0$ and $l/m=\epsilon$. The last but one term of the right-hand side of (18) then becomes

$$\frac{\epsilon\sigma^2}{G}\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}(r^2\delta r)dm = -\frac{\epsilon\sigma^2}{4\pi G}\int_0^M \frac{(\delta\rho)^2}{\rho^3}dm.$$

The last term in the same expression is negative for any model possessing convective stability, for the condition for convective stability can be put in the form

$$-\frac{1}{4(\gamma-1)p'}\frac{dp'}{dm} < -\frac{1}{\gamma p}\frac{dp}{dm},$$

or, using (8),

$$\frac{\kappa l}{64\pi^2 cr^4(\gamma-1)p'} < \frac{Gm}{4\pi r^4\gamma p}.$$

Hence every term on the right-hand side of (18) is negative, and there is stability, provided that $s(\gamma-1)-n$ is positive or zero; this is satisfied for the opacity laws $\kappa=$ const. (the standard model *) and $\kappa \propto \rho/T^{7/2}$. For high-frequency oscillations both the last but one term and (on account of the large number of nodes) the last term will be larger in proportion to the others than for low-frequency oscillations, confirming the suggestion made above on general grounds that the flow of heat tends to make the higher-frequency oscillations more stable than the lower.† It appears, therefore, that if, in the general case, the principal oscillation gives stability, we may infer that the star is stable for any disturbance.

6. Consider, then, the principal oscillation, for which δp has at any given time the same sign for all values of m. Take this sign as negative; then $\delta\rho < 0$, and so

$$\frac{d}{dm}(r^2\delta r) > 0,$$

and as $r^2\delta r = 0$ if $m=0$, $\delta r > 0$ when $m > 0$.

* A similar result follows by the same method for the standard model if the effect of radiation pressure is taken accurately into account.

† This result has been confirmed in one special case by the work of Edgar, to which reference was made above.

Now, by (4), (8) and (9),

$$\frac{d}{dm}(r^3\delta p + 3\gamma p r^2 \delta r) = \frac{1}{4\pi}\left\{\frac{3\delta p}{\rho} + r\delta r\left(\sigma^2 + \frac{4Gm}{r^3}\right) - 3\gamma\left(\frac{Gm\delta r}{r^2} + \frac{p\delta\rho}{\rho^2}\right)\right\}$$

$$= \frac{r\delta r}{4\pi}\left(\sigma^2 - (3\gamma - 4)\frac{Gm}{r^3}\right).$$

Integrate with respect to m over the whole mass. The left-hand side of the resulting equation vanishes ; thus

$$\int_0^M r\delta r\left(\sigma^2 - (3\gamma - 4)\frac{Gm}{r^3}\right)dm = 0. \tag{19}$$

But $r\delta r > 0$, and m/r^3 steadily decreases as m increases. Hence for this result to hold, $\sigma^2 - (3\gamma - 4)Gm/r^3$ must increase from negative to positive values as m increases from 0 to M ; thus $r^3\delta p + 3\gamma p r^2 \delta r$ must first decrease, then increase. But it vanishes when $m = 0$ and $m = M$, and so for intermediate values it must be negative. Thus

$$\frac{\delta\rho}{\rho} + \frac{3\delta r}{r} = \frac{\delta p}{\gamma p} + \frac{3\delta r}{r} < 0. \tag{20}$$

We therefore have, using (4) and (9), that

$$\frac{d}{dm}\left(\frac{\delta r}{r}\right) = \frac{d}{dm}\left(\frac{4\pi r^2 \delta r}{4\pi r^3}\right) = -\frac{\delta\rho}{4\pi\rho^2 r^3} - \frac{3\delta r}{4\pi\rho r^4} > 0.$$

Hence $\delta r/r$ increases steadily with m.

Again, by (8) and (9),

$$-\frac{d\delta p}{dp} = \frac{\delta r}{r}\left(4 + \frac{\sigma^2 r^3}{Gm}\right)$$

and both factors on the right-hand side steadily decrease as m decreases, *i.e.* as p increases. But $\delta p = 0$ when $p = 0$; thus $-\delta p/p$ steadily decreases as p increases, *i.e.* steadily increases with m. It appears probable from the results of numerical integrations effected that not only $\delta r/r$ and $-\delta p/p$, but also their derivatives with respect to r, increase steadily with m.

It also follows from (19) that

$$(3\gamma - 4)G(m/r^3)_c > \sigma^2 > (3\gamma - 4)GM/R^3, \tag{21}$$

where the suffix c refers to the centre of the model. Since the integration in (19) is with respect to the mass, the central regions, where most mass is, will contribute most heavily to the integral, and so σ^2 will be closer to its upper limit than to its lower. The closer $(3\gamma - 4)Gm/r^3$ is to being constant, the closer does the inequality (20) come to being replaced by an equality, and the more nearly do $\delta r/r$, $\delta p/p$ approach constancy. Thus Jeans's assumption is nearly realised if γ is nearly 4/3 or if the density is fairly uniform.

7. Let us apply these results to equation (18). The last term on the

right-hand side we have already shown to be negative. The last but one term is

$$\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{l\sigma^2r^2\delta r}{Gm}\right)dm = -\frac{\sigma^2}{G}\int_0^M \frac{\delta\rho}{\rho}\left\{\frac{l\delta\rho}{4\pi m\rho^2}+r^2\delta r\frac{d}{dm}\left(\frac{l}{m}\right)\right\}dm.$$

But l/m may be expected to be a steadily decreasing function of m, and so, by (20), this expression is less than

$$-\frac{\sigma^2}{G}\int_0^M\left(\frac{\delta\rho}{\rho}\right)^2\left\{\frac{l}{4\pi m\rho}+\frac{r^3}{3}\frac{d}{dm}\left(\frac{l}{m}\right)\right\}dm = -\frac{\sigma^2}{3G}\int_0^M\left(\frac{\delta\rho}{\rho}\right)^2\frac{d}{dm}\left(\frac{lr^3}{m}\right)dm,$$

and, as l and r^3/m both increase steadily with m, this can be set equal to

$$-\frac{\sigma^2\Delta}{3G}\int_0^M\frac{d}{dm}\left(\frac{lr^3}{m}\right)dm = -\frac{\sigma^2LR^3\Delta}{3GM},$$

where Δ is a mean value of $(\delta\rho/\rho)^2$, so weighted that regions in which lr^3/m increases most affect the mean value most. Since the major part of the energy is generated in the neighbourhood of the centre, near the centre lr^3/m increases from zero to approximately $L(r^3/m)_c$, and in the remainder of the star increases to LR^3/M, which is many times as big (54·36 times for the standard model); of the latter increase the major part occurs in the outer regions of the star, where $(\delta\rho/\rho)^2$ is larger than in the central regions. In the mean value Δ', defined by

$$L\Delta' = \int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

on the other hand, the value of $(\delta\rho/\rho)^2$ in the outer regions makes only a small contribution, as little energy is generated in these regions. We may therefore take Δ as greater than Δ'. Hence

$$\begin{aligned}\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{l\sigma^2r^2\delta r}{Gm}\right)dm &< -\frac{\sigma^2R^3}{3Gm}\int_0^M\epsilon\left(\frac{\delta\rho}{\rho}\right)^2dm\\ &< -\frac{(3\gamma-4)}{3}\int_0^M\epsilon\left(\frac{\rho}{\delta\rho}\right)^2dm\end{aligned}$$

using (21).

Again

$$L\left(\frac{\delta\rho}{\rho}\right)_M^2 > \int_0^M\epsilon\left(\frac{\delta\rho}{\rho}\right)^2dm.$$

On use of these results in (18) we have

$$\int_0^M\frac{\delta\rho}{\rho}\left(\delta\epsilon+\frac{d\delta F}{dm}\right)dm < \{\alpha+n+(\beta-s)(\gamma-1)-\tfrac{4}{3}(3\gamma-4)\}\int_0^M\epsilon\left(\frac{\delta\rho}{\rho}\right)^2dm,$$

and so stability will be possible for certain cases which do not satisfy Jeans's criterion given in (2).

8. Consider, for example, the case in which $\kappa \propto \rho/T^{7/2}$. For convective stability the polytropic index of the material, defined as equal to

$$1/\left\{\frac{d\log p}{d\log \rho}-1\right\},$$

cannot be less than $1/(\gamma-1)$, while in the outer layers it has a value between $3\frac{1}{4}$ and 7, according to the magnitude of the ratio of gas pressure to radiation pressure.*

Suppose first that $\gamma = 5/3$. Then the polytropic index cannot fall below 3/2 and the variation of $\delta\rho/\rho$ between centre and boundary, and, in consequence, the degree of stability of the model will be underestimated if we suppose it is equal to 3/2 throughout, since the degree of condensation towards the centre increases steadily with polytropic index.

A numerical integration has been effected for the principal oscillation of a polytrope of index 3/2, setting $\gamma = 5/3$ in equation (17). If for the variables r, p, ρ are substituted their values in terms of the variables u, z used by Eddington,† equations (9) reduce to

$$\frac{d}{dz}\left(\frac{\delta\rho}{\rho}\right) = -\frac{3\Sigma^2 z}{2u}\frac{\delta r}{r} - \frac{1}{u}\frac{du}{dz}\left(\frac{5}{2}\frac{\delta\rho}{\rho} - \frac{6\delta r}{r}\right),$$

$$\frac{d}{dz}\left(\frac{\delta r}{r}\right) = \frac{1}{z}\left(\frac{\delta\rho}{\rho} - \frac{3\delta r}{r}\right),$$

where $\Sigma^2 = \sigma^2/4\pi G\rho_c = (\sigma^2/3G)(r^3/m)_c$. The solution proceeds by a numerical integration of these equations, beginning with $z = 0$, where we set $\delta r/r = 1$, and integrating outwards (the variable z is proportional to r). The values of u, du/dz are taken from the table given by Fairclough ‡; in this z has the value 3·654 at the boundary, and

$$R^3/M = 5\cdot991(r^3/m)_c.$$

The value of Σ^2 is found to be approximately 3/20, corresponding to

$$\sigma^2 = \frac{9}{20}\left(\frac{Gm}{r^3}\right)_c = 2\cdot696\frac{GM}{R^3}.$$

The values of $\delta r/r$, $\delta\rho/\rho$ for different values of z are set out in the table. Corresponding to $z = 3$ the solution begins, in Eddington's words, to get out of hand, showing that this value of σ^2 is a little too small. A second solution, calculated with $\Sigma^2 = 1/6$, on the other hand, begins to "go wrong" in the opposite direction when $z = 2$, so that the error can only be small. We observe that $\delta\rho/\rho$ is about twice as large at the boundary as at the centre.

Of the mass of the model rather more than half lies closer to the centre than the layer given by $z = 2$; the sources of energy will be concentrated

* Jeans, *Astronomy and Cosmogony*, p. 88.
† *The Internal Constitution of the Stars*, chap. iv.
‡ Table I in *M.N.*, **92**, 644, 1932.

z	$\delta r/r$	$-3\delta\rho/\rho$
0	1	3
0·2	1·001	3·006
0·4	1·004	3·023
0·6	1·010	3·052
0·8	1·018	3·093
1·0	1·028	3·146
1·2	1·042	3·214
1·4	1·058	3·297
1·6	1·077	3·398
1·8	1·100	3·519
2·0	1·126	3·663
2·2	1·157	3·834
2·4	1·193	4·034
2·6	1·234	4·271
2·8	1·281	4·552
3·0	1·335	4·888
3·2	1·399	5·300

still closer to the centre, so the mean value Δ' introduced in the last section may be estimated as given by

$$\Delta' \equiv \frac{1}{L}\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm = (1\cdot2)^2\left(\frac{\delta\rho}{\rho}\right)_c^2.$$

Also the ratios of the values of r^3/m at $z=1$, $z=2$ and $z=3$ to its value at the centre are 1·160, 1·697 and 3·518; thus the mean value Δ may be estimated as given by

$$\Delta \equiv \frac{M}{LR^3}\int_0^M \frac{d}{dm}\left(\frac{lr^3}{m}\right)\left(\frac{\delta\rho}{\rho}\right)^2 dm = (1\cdot5)^2\left(\frac{\delta\rho}{\rho}\right)_c^2.$$

These results give us that

$$L\left(\frac{\delta\rho}{\rho}\right)_M^2 = 2\cdot78\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm$$

$$\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{l\sigma^2r^2\delta r}{Gm}\right)dm < -\frac{\sigma^2LR^3\Delta}{3GM} = -1\cdot404\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm.$$

Using these values in (18) we have

$$\begin{aligned}\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d\delta F}{dm}\right)dm &< (\alpha+\beta(\gamma-1)-1\cdot89(s(\gamma-1)-n+3\gamma-4)-1\cdot404)\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm \\ &< (\alpha+\tfrac{2}{3}\beta-5\cdot81)\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,\end{aligned}$$

so that for a law of generation of energy of the form $\epsilon=\epsilon_0T^\beta$ there is vibrational stability if $\beta<8\cdot7$, which is certainly true if the model possesses convective stability. Since the degree of stability is considerably underestimated by supposing that the polytropic index is equal to 3/2 throughout,

we may safely conclude that, if $\gamma = 5/3$, all models which are convectively stable are also stable for vibrations.

Consider next the case $\gamma = 10/7$. For this the polytropic index ranges between a value not less than $2\frac{1}{3}$ and a value not less than $3\frac{1}{4}$; some idea as to its properties may therefore be obtained by a study of a polytrope of index 3. Numerical results for this case are given by Eddington in the paper on Cepheid variation already quoted,* but these stop far short of the boundary, and so do not give a reliable value of $(\delta\rho/\rho)_M$.

We therefore proceed as follows. At the centre of the model

$$-\left(\frac{\delta\rho}{\rho}\right)_c = 3\left(\frac{\delta r}{r}\right)_c,$$

at the boundary

$$-\gamma\left(\frac{\delta\rho}{\rho}\right)_M = -\left(\frac{\delta p}{p}\right)_M = -\left(\frac{d\delta p}{dp}\right)_M = \left(4 + \frac{\sigma^2 R^3}{GM}\right)\left(\frac{\delta r}{r}\right)_M.$$

But, using Eddington's values, in this case $\sigma^2R^3/GM = 3{\cdot}5$, and $\delta r/r$ is probably about 2·5 times as large at the boundary as at the centre : thus, roughly,

$$\left(\frac{\delta\rho}{\rho}\right)_M = 4{\cdot}4\left(\frac{\delta\rho}{\rho}\right)_c.$$

The estimates of Δ, Δ' made in the same way as above are

$$\Delta' = (1{\cdot}25)^2(\delta\rho/\rho)_c^2, \qquad \Delta = 4(\delta\rho/\rho)_c^2.$$

Hence we obtain on use of these results in (18)

$$\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d\delta F}{dm}\right)dm < (\alpha + \tfrac{3}{7}\beta - 8{\cdot}25)\int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

so that there is vibrational stability if $\epsilon = \epsilon_0 T^\beta$ where $\beta < 19$. We may therefore again conclude that if there is convective stability there is also vibrational stability.

A similar calculation, using Eddington's tables for the case $\gamma = 40/29$, gives that there is stability if

$$\alpha + \tfrac{11}{29}\beta < 2{\cdot}15,$$

which, when $\alpha = 0$, reduces to

$$\beta < 5{\cdot}6.$$

As the bounds of convective stability are much stricter with this value of γ than those given in § 3, it is probable that in this case too there is vibrational stability. For values of γ yet closer to 4/3, however, it appears that models may be convectively stable without also being vibrationally stable.

9. We conclude, then, that unless the ratio of specific heats of the stellar material (gas and radiation) is less than 1·38, a star will be vibrationally stable if the transport of energy is radiative. Two other possibilities are, however, open ; part of the flux of energy may be due to conduction,† either

* Also given in *The Internal Constitution of the Stars*, p. 189.

† See Biermann, *loc. cit.* ; Flügge, *Zeit. für Astrophys.*, **6**, 272, 1933.

by electrons or by neutrons ; this will increase the range of models consistent with either convective or vibrational stability. Also it is not necessary for a star to continue to exist that it should be stable for convection, and indeed, if Atkinson's hypothesis of generation by nuclear penetration is correct, every star must possess a core in which there are convection currents. Further investigation on these lines is required.

We may observe finally that our results show that the condition of vibrational stability is much less strict than Jeans's condition (2). In all the models we have considered $\delta r/r$ varies considerably with r. In consequence the right-hand side of (2) has, in certain cases, to be increased more than ninefold.

Summary.—The problems of the convective and the vibrational stability of a star are discussed. It is found that with Kramers's opacity law convection will appear in a star in which the generation of energy per unit mass varies more rapidly than the eighth power of the temperature, and that the radial oscillations of a star in which convection does not appear will not increase indefinitely unless the ratio of specific heats of the material is less than 1·38.

Swansea :
1934 *May* 31.

NOTE ON THE POSITION OF THE APEX AS DERIVED FROM THE RADCLIFFE STARS OF LARGE PROPER MOTION IN THE SELECTED AREAS.

H. G. Scott Barrett.

(Communicated by the Radcliffe Observer)

A provisional list of stars of large proper motion (0″·096 and greater) as derived from the Radcliffe photographs has already been published in the *M.N.*, **91**, 933, 1931. This list has been extended to all the northern fields (Areas 1–115) of the Systematic Plan of Kapteyn's Selected Areas, and in its completed form will be found in the forthcoming Radcliffe Catalogue of Proper Motions. The proper motions have now been made absolute by the application of the secular parallax of the comparison stars, assuming a Solar Apex at 273°, +44°.

It seemed that a result of interest would be the determination of the Apex of the motions of these stars. This has been carried through by Airy's method, and from all the stars, 333 in number, the position obtained is

$$283°{\cdot}1 \pm 5°{\cdot}0, \quad +37°{\cdot}9 \pm 3°{\cdot}5 ; \qquad p = 0''{\cdot}126 \pm ''{\cdot}008,$$

where p denotes the parallactic motion.

Since, by the method adopted, stars of very large motion must have a preponderating influence on the result obtained, a further solution was made omitting the eight stars which have proper motions exceeding 0″·5

per annum. The position obtained for the Apex is not greatly changed, the principal result, as was to be expected, being a decided reduction in the probable errors. The position now becomes

$$286°\cdot6 \pm 3°\cdot0, \quad +36°\cdot4 \pm 2°\cdot2\,; \qquad p = 0''\cdot117 \pm ''\cdot0045.$$

If the stars are divided into two groups, one of 164 stars of magnitude 12·6 and brighter, and the other of 161 stars fainter than 12·6 mag., we find for the brighter stars ($\bar{m} = 10\cdot7$)

$$286°\cdot8 \pm 3°\cdot6, \quad +33°\cdot8 \pm 2°\cdot6\,; \qquad p = 0''\cdot126 \pm ''\cdot006,$$

and for the fainter stars ($\bar{m} = 14\cdot3$)

$$287°\cdot1 \pm 5°\cdot2, \quad +39°\cdot4 \pm 3°\cdot7\,; \qquad p = 0''\cdot108 \pm ''\cdot007.$$

As the mean value of a proper motion under consideration here is large in proportion to the quantities involved in the reduction to absolute, any change resulting from the assumption of a different position for the Solar Apex for the comparison stars will be quite small. The final result will in any case be affected by only a small fraction of its probable error. If, for instance, the declination of the Apex had been taken as $+34°$ instead of $+44°$, the consequence would be to produce values for the declination of the apex of these stars about $0°\cdot3$ lower than those actually obtained.

Radcliffe Observatory, Oxford :
1934 *May* 31.

THE PROPER MOTIONS OF STARS IN THE REGION OF CLUSTER M13 (N.G.C. 6205).

Violet F. White, M.Sc.

Two plates, with exposures of one hour, centred on the globular cluster in Hercules, M13 ($16^h\ 39^m$, $36°\ 39'$; 1900), taken with the Wilson reflector of the University of London Observatory with an interval of 36 years, were measured for proper motions. The plates were of very good quality and showed well-defined circular images for stars of photographic magnitudes 10 to 15.

The plates were measured separately as in Region IV,* and the measurements combined to give differential values M_x, M_y, from which proper motions μ_x, μ_y were determined from the equations

$$\mu_x = M_x + a + bx + cy, \qquad \mu_y = M_y + d + ex + fy.$$

The plate constants were calculated by Christie and Dyson's method from 36 stars of mean photographic magnitude 14·5 distributed uniformly within a distance of 15′ from the plate-centre. The cluster was of radius 7′, and about half of the comparison stars were within the cluster area.

* *M.N.*, **93**, 647, 1933.

The plates were measured twice in a direct and twice in a reverse position, and two values of the plate constants obtained from the different positions of the plates. The proper motions discussed here are the means of the two reductions.

The probable errors of the mean values calculated from the deviations of the separate determinations from their means were $\pm 0''\cdot3$ per century in both μ_x and μ_y.

The diameters of all the stellar images were measured and the corresponding magnitudes determined as follows: Certain cluster stars were identified with those whose magnitudes have been determined by H. Shapley,* and the diameters corresponding to Mt. Wilson photographic magnitudes 15·1 to 12·5 were thus obtained. The diameters corresponding to 7 B.D. stars of about magnitude 9 were also known. All the magnitudes were converted to the Harvard visual scale by applying a correction given by A. Pannekoek † to the B.D. magnitudes, and by applying a correction ‡

$$C_p = -0\cdot18 + 0\cdot071 m_p$$

for colour-index to the Mt. Wilson photographic magnitudes, converting them to Mt. Wilson photovisual magnitudes, which are the same as Harvard visual magnitudes within the magnitude range considered.§ Diameters corresponding to visual magnitudes 14·2 to 11·8 and 9·7 to 8·5 were thus obtained, and a smooth curve was completed showing the diameters corresponding to the intervening magnitudes.

The mean visual magnitude of 141 cluster stars was 13·5, and 104 stars outside the cluster were divided into magnitude groups 14·2 to 12·7 and 12·6 to 9·8. The proper motions of the 11 stars in the region brighter than 9·8 were not considered.

The proper motions relative to the comparison stars of the cluster stars and the stars outside the cluster are plotted in figs. 1 and 2, both on the same scale and referred to the same origin O.

To convert the relative proper motions to absolute proper motions the mean parallaxes and mean relative proper motions of groups of stars were compared, the values in seconds of arc per century being given in Table I.

TABLE I

Region	Vis. Mag.	No. of Stars	H_m	P_x	$\bar{\mu}_x$	$P_x-\bar{\mu}_x$	P_y	$\bar{\mu}_y$	$P_y-\bar{\mu}_y$
			"	"	"	"	"	"	"
$V_{13\cdot5}$	13·5	67	0·5	−0·1	−0·3	0·2	0·0	−0·4	0·4
$V_{11\cdot0}$	11·0	37	1·1	−0·3	−0·1	−0·2	0·0	0·2	−0·2
V	12·6	104	0·7	−0·2	−0·2	0·0	0·0	−0·2	0·2
Comparison stars	13·6	36	0·5	−0·1	0·1	−0·2	0·0	0·0	0·0
Cluster stars	13·5	141	...	...	0·0	...	...	−0·1	...

* *Contr. Mt. Wilson Obs.*, No. 338, 6, 1927; *Ap. J.*, **66**, 95, 1927.
† *Pubn. Astr. Inst. Univ. Amsterdam*, **1**, Table 26, 1924.
‡ *Contr. Mt. Wilson Obs.*, No. 287, 1924; *Ap. J.*, **61**, 126, 1925.
§ *Contr. Mt. Wilson Obs.*, No. 288, 1924; *Ap. J.*, **61**, 289, 1925.

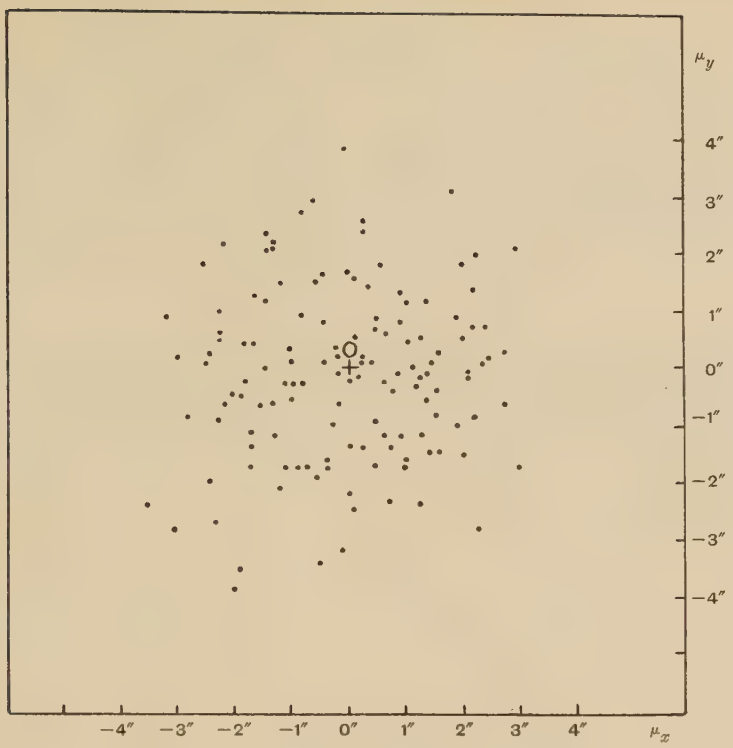

FIG. 1.—*Centennial proper motions of* 141 *cluster stars. Mean vis. mag.* 13·5.

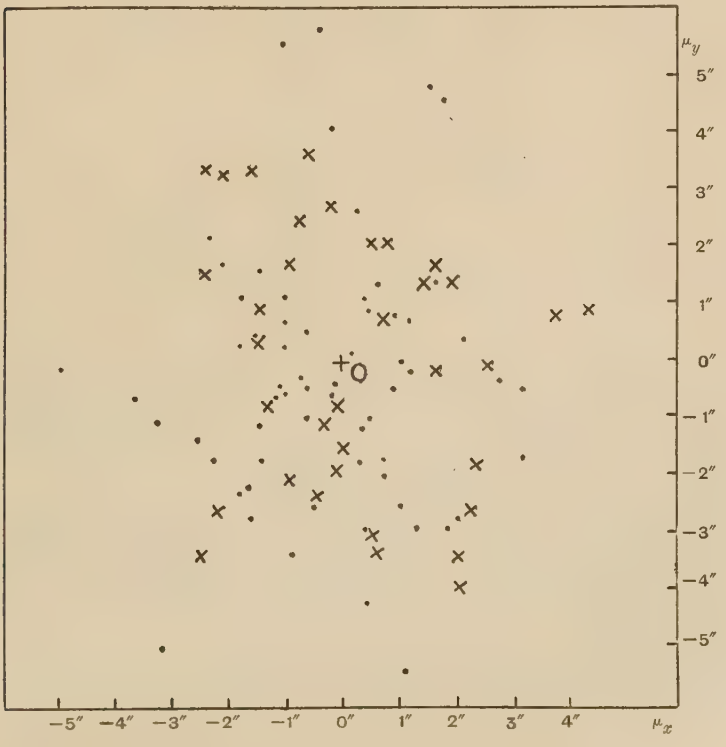

FIG. 2.—*Centennial proper motions of* 104 *stars outside the cluster.*
Dots refer to stars with mean vis. mag. 13·5.
Crosses ,, ,, ,, ,, 11·0.

The correction, obtained from the comparison stars, to be added to the relative proper motions is $-0''\cdot2$, $0''\cdot0$, which is so small that the origin, A, of the absolute proper motions practically coincides with O, the origin of relative proper motions.

With this correction the absolute centennial proper motion of the cluster is

$$\mu_\alpha = -0''\cdot2, \qquad \mu_\delta = -0''\cdot1.$$

If the absolute proper motion of the cluster is assumed to be zero, then the correction to be applied to the relative proper motions is $0''\cdot0$, $0''\cdot1$, again showing that the origin O can be taken as the origin of absolute proper motions.

The numbers of stars moving in the directions of various position angles are given in Table II.

TABLE II

θ	$V_{13\cdot5}$	$V_{11\cdot0}$	V	Cluster	θ	$V_{13\cdot5}$	$V_{11\cdot0}$	V	Cluster
	r					r			
°					°				
0	2	1	3	5	180	0	2	2	5
10	1	1	2	2	190	1	2	3	3
20	4	1	5	1	200	4	1	5	4
30	1	0	1	4	210	3	1	4	5
40	1	2	3	4	220	3	2	5	1
50	2	2	4	5	230	3	0	3	5
60	1	0	1	4	240	3	1	4	4
70	0	0	0	5	250	3	0	3	6
80	1	2	3	4	260	1	0	1	5
90	1	1	2	7	270	1	0	1	5
100	3	1	4	6	280	1	1	2	5
110	0	0	0	4	290	2	1	3	3
120	2	0	2	4	300	2	2	4	1
130	0	1	1	5	310	4	0	4	3
140	1	1	2	2	320	2	0	2	3
150	2	2	4	7	330	0	3	3	6
160	6	0	6	2	340	0	2	2	2
170	5	3	8	2	350	1	1	2	2

From these numbers a drift curve was drawn for all the stars outside the cluster with magnitudes 9·8 to 14·2 (fig. 3). The directions of Drifts I and II and of the solar antapex according to their positions assumed in *M.N.*, **92**, 522, 1932, are indicated in the figure.

The distribution of proper motions in the cluster is approximately circular, and an upper limit for the probable error of the proper motions can be obtained by considering that these internal proper motions represent errors of measurement only. The probable error thus determined is $\pm 0''\cdot9$ per century.

The numbers of stars with centennial proper motions between various limits are given in Table III.

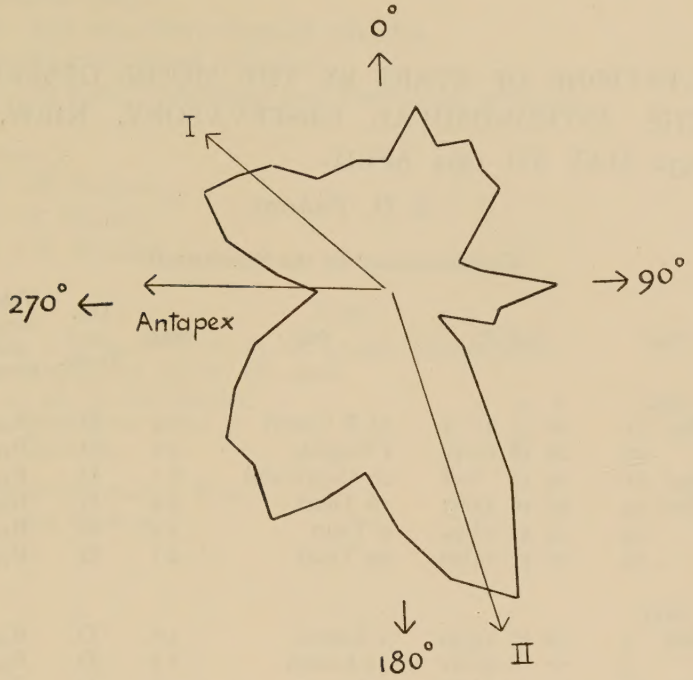

FIG. 3.—*Region V.* 104 *stars.* *Mean vis. mag.* 12·6.

TABLE III

μ	$V_{13\cdot5}$	$V_{11\cdot0}$	V	Cluster
	No. of Stars			
″ ″				
0·0 – < 1·0	6	1	7	23
1·0 – < 2·0	23	9	32	52
2·0 – < 3·0	16	11	27	47
3·0 – < 4·0	13	10	23	15
4·0 – < 5·0	5	5	10	4
⩾5·0	4	1	5	0

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OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE ASTRONOMICAL OBSERVATORY, KIEW, FROM 1932 MAY TO 1934 APRIL.

S. D. Tscherny.

(Communicated by the Secretaries)

No.	Date	G.C.T.	Star	Mag.	Dis. or Re-ap.	Telescope and Power	Observer
	1932	h m s					
1	May 11	20 33 41·33	35 B Cancri	6·4	D	R_{145}	T
2	17	20 38 17·29	*i* Virginis	5·7	D	R_{145}	T
3	Sept. 11	19 32 8·28	27 Capricorni	6·1	D	R_{145}	T
4	Nov. 14	23 16 52·83	16 Tauri	5·4	D	R_{145}	T
5	14	23 34 28·94	9 Tauri	4·3	D	R_{145}	T
6	14	23 57 14·20	20 Tauri	4·1	D	R_{145}	T
	1933						
7	May 3	18 27 15·59	ψ Leonis	5·6	D	R_{145}	T
8	5	21 4 50·22	79 Leonis	5·5	D	R_{145}	T
9	June 29	19 19 42·07	*v* Leonis	4·5	D	R_{145}	T
10	Aug. 2	19 30 56·39	210 B Scorpii	5·8	D	R_{145}	T
11	Sept. 26	16 55 57·27	58 G Sagittarii	6·1	D	R_{145}	T
	1934						
12	Apr. 16	17 46 24·97	16 Tauri	5·4	D	R_{145}	T
13	16	17 46 25·88	16 Tauri	5·4	D	U_{74}	I
14	16	17 46 24·61	16 Tauri	5·4	D	B_{61}	R
15	16	17 55 18·05	B.D. +23° 508	8·9	D	R_{145}	T
16	16	18 1 59·38	9 Tauri	4·3	D	U_{74}	I
17	16	18 2 12·44	17 Tauri	3·8	D	R_{145}	T
18	16	18 2 12·78	17 Tauri	3·8	D	B_{61}	R
19	16	18 8 20·45	B.D. +23° 512	8·1	D	R_{145}	T
20	16	18 8 20·88	B.D. +23° 512	8·1	D	U_{74}	I
21	16	18 8 20·61	B.D. +23° 512	8·1	D	B_{61}	R
22	16	18 10 55·04	B.D. +24° 550	8·8	D	R_{145}	T
23	16	18 13 42·59	20 Tauri	4·1	D	R_{145}	T
24	16	18 13 42·63	20 Tauri	4·1	D	U_{74}	I
25	16	18 13 42·23	20 Tauri	4·1	D	B_{61}	R
26	16	18 27 57·31	21 Tauri	5·8	D	R_{145}	T
27	16	18 27 57·61	22 Tauri	6·5	D	R_{145}	T
28	16	18 27 57·88	22 Tauri	6·5	D	U_{74}	I
29	16	18 31 27·60	B.D. +23° 523	8·0	D	B_{61}	R
30	16	18 31 51·82	B.D. +23° 520	8·1	D	B_{61}	R
31	16	18 43 31·81	16 Tauri	5·4	R	R_{145}	T
32	16	18 52 18·41	9 Tauri	4·3	R	R_{145}	T
33	16	19 3 42·58	21 Tauri	5·8	R	R_{145}	T
34	16	19 10 15·54	22 Tauri	6·5	R	R_{145}	T
35	18	19 15 26·11	112 B Aurigæ	5·7	D	R_{145}	T
36	18	19 15 26·08	112 B Aurigæ	5·7	D	B_{61}	R

Instruments used :

R = 243 mm. Merz-Repsold refractor.
B = 83 mm. Busch refractor.
U = 72 mm. Utzscheider and Fraunhofer refractor.

Observers :

T = S. Tscherny.
I = I. Iljinsky.
R = N. Rudzky.

Notes

Nos. 2, 3, 7, 8, 9, 12, 16, 17, 23, 26, 27, 35 : instantaneous.
Nos. 5, 6, 10, 11, 15, 19, 22, 32 : good.
Nos. 1, 4, 31, 33 : satisfactory.
No. 34 : retardation about 1^s.
No. 13 : uncertain.

Astronomical Observatory, Kiew :
1934 April 24.